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Maximum of the Membrane Model on Regular Trees

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Abstract

The discrete membrane model is a Gaussian random interface whose inverse covariance is given by the discrete biharmonic operator on a graph. In literature almost all works have considered the field as indexed over $\mathbb{Z}^d$, and this enabled one to study the model using methods from partial differential equations. In this article we would like to investigate the dependence of the membrane model on a different geometry, namely trees. The covariance is expressed via a random walk representation which was first determined by Vanderbei in (Ann Probab 12:311–314, 1984). We exploit this representation on m-regular trees and show that the infinite volume limit on the infinite tree exists when $m \geq 3$. Further we determine the behavior of the maximum under the infinite and finite volume measures.

Keywords Random interfaces · Membrane model · Trees · Extremes · Random walk representation

Mathematics Subject Classification 60G15 · 82B20 · 82B41 · 60G70

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1 Introduction

The main object of study in this article is the membrane model (MM), also known as discrete bilaplacian or biharmonic model. As a random interface, the MM can be defined as a collection of Gaussian heights indexed over a graph. In this article, we will study the MM on regular trees. Let $\mathbb{T}_m$ be an m-regular infinite tree, that is, a rooted tree with the root having m-children and each of the children thereafter having $m-1$ children. With abuse of notation we will denote the vertex set of $\mathbb{T}_m$ by $\mathbb{T}_m$ itself. Then the MM is defined to be a Gaussian field $\varphi = (\varphi_x)_{x\in\mathbb{T}_m}$, whose distribution is determined by the probability measure on $\mathbb{R}^{\mathbb{T}_m}$ with density

$$\mathbf{P}_\Lambda(\mathrm{d}\varphi) := \frac{1}{Z_\Lambda}\exp\left(-\frac{1}{2}\sum_{x\in\mathbb{T}_m}(\Delta\varphi_x)^2\right)\prod_{x\in\Lambda}\mathrm{d}\varphi_x\prod_{x\in\mathbb{T}_m\setminus\Lambda}\delta_0(\mathrm{d}\varphi_x). \tag{1.1}$$

Here $\Lambda\subset\mathbb{T}_m$ is a finite subset, Δ is the discrete Laplacian defined by

$$\Delta f_x := \Delta f(x) := \sum_{y\sim x}\frac{1}{m}(f(y)-f(x)), \quad f:\mathbb{T}_m\to\mathbb{R},\ x\in\mathbb{T}_m, \tag{1.2}$$

where $y\sim x$ means that y is a neighbor of x, $\mathrm{d}\varphi_x$ is the Lebesgue measure on $\mathbb{R}$, δ_0 is the Dirac measure at 0, and Z_Λ is a normalising constant. We are imposing zero boundary conditions i.e. almost surely $\varphi_x = 0$ for all $x\in\mathbb{T}_m\setminus\Lambda$, but the definition holds for more general boundary conditions.

The membrane model was introduced and studied mostly in the case $\Lambda\subset\mathbb{Z}^d$. For example, the existence of an infinite volume measure for $d\geq 5$ was proved in [16] and later the model and its properties were studied in details in [13]. The point process convergence of extremes on $\mathbb{Z}^d$ for $d\geq 5$ was dealt with in [7]. The case of $d=4$ is related to log-correlated models and the limit of the extremes was derived in [17]. Finally the scaling limit of the maximum in lower dimensions follows from the scaling limit of the model which was obtained by [6] in $d=1$ and by [8] in $d=2,\ 3$.

The discrete Gaussian free field (DGFF) is a well studied example of a discrete interface model and has connections to other stochastic processes, such as branching random walk and cover times. Most of these connections arise due to the fact that the covariance of the DGFF is the Green's function of the simple random walk. This is not the case for the MM, essentially because the biharmonic operator does not satisfy a maximum principle. This also depends heavily on the boundary conditions: closed formulas for the bilaplacian covariance matrix have been found [10, 11, 13], however they do not apply to our choice of boundary values. On the square lattice one can rely on other techniques, namely discrete PDEs, to prove results in the bilaplacian case. However as soon as one goes beyond $\mathbb{Z}^d$ approximations of boundary value problems are less straightforward, and our work is prompted from this aspect. We will use a probabilistic solution of the Dirichlet problem for the bilaplacian [19] to investigate the membrane model indexed on regular trees. We restrict our study to regular trees because these graphs have many features which are different from $\mathbb{Z}^d$. One of the most striking contrasts is that the number of vertices in the n-th generation is comparable to the size of the graph up to the n-th generation. From Vanderbei's representation, it is clear that the boundary plays a prominent role in the behavior of the covariance structure. We will use this representation to derive the maximum of the field under the infinite and finite volume measures. In the next section we describe our set-up and also state the main results, followed by a discussion on future directions.

2 Main Results

2.1 The Model

For any two vertices $x, y \in \mathbb{T}_m$, we denote $d(x, y)$ to be the graph distance between x and y. Then the Laplacian, whose definition was given in (1.2), can also be viewed as the following matrix:

$$\Delta(x, y) = \begin{cases} -1 & \text{if } x = y, \\ \frac{1}{m} & \text{if } d(x, y) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We write Δ^2 for its iteration, i.e., $\Delta^2 f_x := \Delta(\Delta f(x))$ and define Δ^2_Λ to be the matrix $(\Delta^2(x, y))_{x,y\in\Lambda}$.

Lemma 1 *The Gibbs measure $\mathbf{P}_\Lambda$ on $\mathbb{R}^\Lambda$ with 0-boundary conditions outside Λ given by (1.1) exists for any finite subset Λ. It is the centered Gaussian field on Λ with covariance matrix $(\Delta^2_\Lambda)^{-1}$.*

Proof We first prove that Δ^2 is symmetric and positive definite, i.e., for any function $f : \mathbb{T}_m \to \mathbb{R}$ which vanishes outside a finite subset and which is not identically zero

$$\sum_{x,y\in\mathbb{T}_m} f(x)\Delta^2(x, y) f(y) > 0. \tag{2.2}$$

From (2.1) it is clear that Δ is symmetric, and hence Δ^2 is so. Let $g = \Delta f$ and to prove (2.2) we observe that

$$\begin{aligned} \sum_{x,y\in\mathbb{T}_m} f(x)\Delta^2(x, y) f(y) &= \sum_{x\in\mathbb{T}_m} f(x)\Delta g(x) = \frac{1}{m}\sum_{x\in\mathbb{T}_m} f(x)\sum_{y\sim x}(g(y) - g(x)) \\ &= \frac{1}{m}\sum_{x\in\mathbb{T}_m} g(x)\sum_{y\sim x}(f(y) - f(x)) = \sum_{x\in\mathbb{T}_m} g(x)g(x) > 0. \end{aligned}$$

Also, one can show using summation by parts that if $\varphi : \mathbb{T}_m \to \mathbb{R}$ vanishes outside Λ then

$$\sum_{x\in\mathbb{T}_m} (\Delta\varphi_x)^2 = \sum_{x\in\mathbb{T}_m} \varphi_x \Delta^2 \varphi_x.$$

The proof is now complete by using Proposition 13.13 of [9]. □

2.2 Main Results

We denote the root of the tree by o. We will consider $m \geq 3$. In the case when $m = 2$ the tree is isomorphic to $\mathbb{Z}$ and the MM on $\mathbb{Z}$ has been studied in the literature, see for instance [5, 6]. For any $n \in \mathbb{N}$, we define

$$V_n := \{x \in \mathbb{T}_m : d(o, x) \leq n\}.$$

Let $\varphi = (\varphi_x)_{x\in\mathbb{T}_m}$ be the membrane model on $\mathbb{T}_m$ with zero boundary conditions outside V_n. In this case, we denote the corresponding measure $\mathbf{P}_{V_n}$ by $\mathbf{P}_n$. Also we denote the covariance function for this model by G_n, that is, $G_n(x, y) := \mathbf{E}_n[\varphi_x\varphi_y]$. Let $(S_k)_{k\geq 0}$ be the simple random walk on $\mathbb{T}_m$. We write $\mathbf{P}_x$ for the canonical law of the simple random walk starting at x. The following theorem proves the existence of the infinite volume limit.

Theorem 2 *The measures* $\mathbf{P}_n$ *converge weakly to a measure* $\mathbf{P}$, *which is the law of a Gaussian process* $(\varphi_x)_{x\in\mathbb{T}_m}$ *with covariance function* G *given by*

$$G(x, y):=\mathbf{E}[\varphi_x\varphi_y] = \mathbf{E}_x\left[\sum_{k=0}^{\infty}(k+1)\,\mathbb{1}_{[S_k=y]}\right] = \sum_{k=0}^{\infty}(k+1)\mathbf{P}_x(S_k=y).$$

We will see later (in Lemma 9) that for any $x \in \mathbb{T}_m$

$$G(x, x) = G(o, o) = \frac{(m-1)((m-1)^2+1)}{(m-2)^3}.$$

We define two sequences as follows

$$b_n:=\sqrt{G(o, o)}\left[\sqrt{2\log N} - \frac{\log\log N + \log(4\pi)}{2\sqrt{2\log N}}\right], \quad a_n:=G(o, o)b_n^{-1},$$

where $N:=|V_n|$. We have

$$N = 1 + \sum_{k=1}^{n} m(m-1)^{k-1} = \frac{m(m-1)^n - 2}{m-2}. \tag{2.3}$$

Our main result in this paper concerns the scaling limit of the maximum of the field, namely the Gumbel convergence of the rescaled maximum.

Theorem 3 *For any* $\theta \in \mathbb{R}$

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{\max_{x\in V_n}\varphi_x - b_n}{a_n} \le \theta\right) = \exp(-\mathrm{e}^{-\theta}).$$

We show in the following result that up to the first order the constants do not change for the extremes and when we look at the expected maximum under the finite volume, it still converges to $\sqrt{G(o, o)}$, after appropriate scaling. The same result can be proved under the infinite volume measure, so we stick to the finite volume case, the infinite volume situation being completely analogous.

Theorem 4 *For* $m \ge 3$,

$$\lim_{n\to\infty}\frac{\mathbf{E}_n\left[\max_{x\in V_n}\varphi_x\right]}{\sqrt{2\log N}} = \sqrt{G(o, o)}.$$

In case of the finite volume field we show that the maximum field normalised to have variance one converges in distribution to the Gumbel distribution. We define $B_n:=b_n/\sqrt{G(o, o)}$ and $A_n:=B_n^{-1}$.

Theorem 5 *Let* $\psi_x = \varphi_x/\sqrt{G_n(x, x)}$ *for* $x \in V_n$. *Then for any* $\theta \in \mathbb{R}$ *and* $m \ge 14$ *we have that*

$$\lim_{n\to\infty}\mathbf{P}_n\left(\frac{\max_{x\in V_n}\psi_x - B_n}{A_n} \le \theta\right) = \exp(-\mathrm{e}^{-\theta}).$$

Remark 6 In exposing our results we keep all the constants depending on m explicit. We emphasize that $m \ge 14$ is just the bound that our approach yields, while presumably the result holds for all $m \ge 3$.

2.3 Discussion

- Our result is heuristically motivated by the fast decay of correlations of the DGFF on a tree. As we shall see in Lemma 9 correlations decay exponentially in the distance between points, which suggests a strong decoupling and a behavior of the rescaled maximum similar to that of independent and identically distributed Gaussians.
- In Theorem 3, the scaling constants show that the correlation structure can be ignored for the extremes and the behaviour is similar to that of i.i.d. centered Gaussian random variables with variance $G(o, o)$. In the finite volume case, we rescaled the field to have variance one and hence the behaviour remains the same as that of the i.i.d. case. An interesting open problem is whether this behavior is retained for the finite volume field divided by $\sqrt{G(o, o)}$. This convergence is by no means trivial to obtain as the finite volume variance convergence to $G(\cdot, \cdot)$ is not uniform, in particular the error depends on the distance to the leaves. Moreover, since the size of the boundary of a finite tree is non-negligible with respect to the total size we cannot claim that extremes are achieved in the bulk and not near the boundary.
- For scaling extremes we use a comparison theorem of [12] that is based on Stein's method. There are many different approaches to the question of convergence of extremes using Stein's method, one notable instance being [2]. Compared to their method the advantage of [12] is that it does not require to control the conditional expectation of the field that emerges from the spatial Markov property. While for other interfaces, like the DGFF, the harmonic extension has a closed form in terms of random walk probabilities, the biharmonic extension is more subtle to bound, and [12] allow one to bypass this step.
- The main contribution of the article is the analysis of the covariance structure for a membrane model on the tree. We are not aware of any prior work which deals with the bilaplacian model on graphs beyond $\mathbb{Z}^d$, whilst there is an extensive literature on the discrete Gaussian free field on general graphs. We exploit the representation of the solution of a biharmonic Dirichlet problem in terms of the random walk on the graph. In the bulk the behaviour is easy to derive and is close to that of

  $$\overline{G}_n(x, y) := \mathbf{E}_x \left[\sum_{k=0}^{\tau_0 - 1} (k+1)\, \mathbb{1}_{[S_k = y]} \right],$$

  where τ_0 is the first exit time from a bounded subgraph (see Sect. 3). Around the boundary additional effects arising out of boundary excursions kick in, in particular we will use the successive excursion times of the random walk and the local time of the random walk between two consecutive visits to the complement of a set. Control of such observables on general graphs will open up avenues for further interesting studies in the area of random interfaces.
- Although we consider regular trees we believe that our results can be extended to rooted trees where the same scaling limit for the maximum will hold. The case of Galton–Watson tree will be more challenging due to the randomness of the offspring distribution, but would be an intriguing direction to extend the study of random interfaces to random graphs.

Structure of the article In Sect. 3 we recall the random walk representation for the solution of the biharmonic Dirichlet problem for a general graph and also rewrite the formula in our set-up. In Sect. 4 we show that the infinite volume membrane measure exists and provide a proof of Theorem 2. In Sect. 5 we prove Theorem 3 providing a limit for the expected maximum under the finite volume measure. In Sect. 6 we use the estimates to determine the

fluctuations of the extremes in the infinite volume and prove Theorem 4. In Sect. 7 we show the fluctuations of the maximum under the finite volume measure. Section 8 is devoted to the proof of Lemma 15 which is related to finer estimates on the covariance of the model.
Notation In the following C is a generic constant which may depend on m and may change in each appearance within the same equation.

3 A Random Walk Representation for the Covariance Function

In this section we shall revisit the random walk representation for the covariance function G_n. From the definition of the model it follows that G_n satisfies the following Dirichlet problem: for $x \in V_n$

$$\begin{cases} \Delta^2 G_n(x, y) = \delta_x(y), & \text{if } y \in V_n \\ G_n(x, y) = 0, & \text{if } y \in V_n^c. \end{cases} \tag{3.1}$$

If one considers the Dirichlet problem above but with $-\Delta$ replacing Δ^2 then the solution is the well-known expected local time of the simple random walk on the graph [18, Chapter 1]. In our set-up such a general easy formulation is not available. In particular, to the best of the authors' knowledge one cannot relate the covariance of the MM to a stochastic process. The solution is then given by a weighted local times and an expression involving the boundary excursion times of the random walk. The boundary effects are more profound in the membrane model and this is documented in the existing works on $\mathbb{Z}^d$ ([8, 14, 15, 17]).

3.1 Intermezzo: Random Walk Representation on General Graphs

In this subsection we discuss the probabilistic solution of the Dirichlet problem for the discrete biharmonic operator obtained by [19], whose set-up is much more general in that it considers general graphs and not only trees. We recall it here for completeness. Let $\mathcal{G}$ be a connected graph and let $\Lambda \subset \mathcal{G}$ be a finite subgraph. With a slight abuse we will confound the graph $\mathcal{G}$ resp. Λ with its vertex set, but this should not cause any confusion. Let ρ be a strictly positive measure on the discrete state space $\mathcal{G}$ and for all $x, y \in \mathcal{G}$, $q(x, y)$ be a positive symmetric transition function such that

$$\sum_{y \in \mathcal{G}} q(x, y)\rho(y) = 1.$$

Let $P = (p(x, y))_{x,y \in \mathcal{G}}$ be a transition matrix such that

$$p(x, y) = q(x, y)\rho(y).$$

Let $(S_k)_{k \geq 1}$ be a random walk on $\mathcal{G}$, defined on a probability space $(\Omega, \mathcal{F})$, with transition matrix P making the random walk symmetric. Now the Laplacian operator acting on a function $f : \mathcal{G} \to \mathbb{R}$ is defined as

$$(\Delta f)(x) = \sum_{y \sim x} p(x, y)(f(y) - f(x)).$$

The one-step transition operator P is defined as

$$(Pf)(x) = \mathbf{E}_x[f(S_1)] = \sum_{y \sim x} p(x, y) f(y).$$

Then

$$\Delta = P - I,$$

where I is the identity operator. We say that f is a solution to the non-homogeneous Dirichlet problem for the bilaplacian if f satisfies the following:

$$\begin{cases} \Delta^2 f(y) = \psi(y), & \text{if } y \in \Lambda \\ f(y) = \phi(y), & \text{if } y \in \partial_2\Lambda, \end{cases} \tag{3.2}$$

where $\partial_k\Lambda$ is defined by

$$\partial_k\Lambda := \{z \in \Lambda^c : d(z, \Lambda) \le k\}, \quad k \ge 1 \tag{3.3}$$

and where ψ, ϕ are graph functions representing the input datum resp. boundary datum. We want to obtain a probabilistic solution of the problem (3.2). We define τ_i to be the $(i+1)$-th visit time to Λ^c by the random walk S_k. Formally,

$$\tau_i := \inf\{k > \tau_{i-1} : S_k \in \Lambda^c\}, \quad \tau_{-1} := -1. \tag{3.4}$$

Note that τ_0 is the first exit time from Λ. We will keep two assumptions throughout the Section:

(i) $\Lambda \cup \partial_1\Lambda$ is finite;
(ii) $\mathbf{E}_x\left[\tau_0^2\right] < \infty$ for all $x \in \mathcal{G}$.

Let

$$L^2(\mathcal{G}, \rho) = \left\{ f \,\middle|\, f : \mathcal{G} \to \mathbb{R} \text{ such that } \sum_{x\in\mathcal{G}} f(x)^2\rho(x) < \infty \right\}$$

and the inner-product in $L^2(\mathcal{G}, \rho)$ be defined as follows: for $f, g \in L^2(\mathcal{G}, \rho)$

$$\langle f, g\rangle_{\mathcal{G}} := \sum_{x\in\mathcal{G}} f(x)g(x)\rho(x).$$

One can show that P is a self-adjoint operator on $L^2(\mathcal{G}, \rho)$. Hence Δ is also self-adjoint. We define

$$\begin{cases} M_{-1} := 1 \\ M_j := \prod_{i=0}^{j} (\tau_i - \tau_{i-1} - 1), \quad j \ge 0. \end{cases}$$

Next we define an operator acting on $\mathcal{A} = \{f \mid f : \Lambda^c \to \mathbb{R}\}$. The operator Q acting on $\mathcal{A}$ is defined as

$$(Qf)(x) = \mathbf{E}_x\left[(\tau_1 - 1) f\left(S_{\tau_1}\right)\right], \quad x \in \Lambda^c. \tag{3.5}$$

Observe that, if $x \in (\Lambda \cup \partial_1\Lambda)^c$, then $Qf(x) = 0$ for all $f \in \mathcal{A}$. Therefore

$$\text{Range}(Q) \subset \left\{ g \,\middle|\, g : \Lambda^c \to \mathbb{R} \text{ and } g(x) = 0, \forall x \notin \partial_1\Lambda \right\} \subset \mathcal{A}.$$

Since $\partial_1\Lambda$ is finite, one can show with the help of (ii) that $(Qf)(x)$ is bounded. It can be shown that the operator Q is positive semi-definite on $L^2(\Lambda^c, \rho)$ (see [19]). Therefore Q can be diagonalized and can be written as

$$Q = \sum_{\lambda} \lambda \Pi_\lambda$$

where the sum is over all the eigenvalues of Q and Π_λ are the projection operators onto the eigenspace corresponding to the eigenvalue λ. Observing the range of the Q-operator in (3.6) and the fact that $\partial_1\Lambda$ is finite, we can say that the operator Q is compact and Range(Q) is finite-dimensional. Therefore, the spectrum of Q is finite. Also, as Q is positive semi-definite, we conclude that all the eigenvalues are non-negative. A probabilistic solution of the problem (3.2) is given in [19].

Theorem 7 ([19, Theorem 4]) *Let $(\eta_t)_{t\geq 0}$ be a Poisson process with parameter 1 which is independent of the random walk (S_k). Then the solution of (3.2) is given by*

$$f(x) = \lim_{t\to\infty} \mathbf{E}_x \left[\sum_{j=0}^{\eta_t} (-1)^j M_{j-1} \left[(\tau_j - \tau_{j-1})\phi(S_{\tau_j}) + \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1})\psi(S_k) \right] \right]. \tag{3.6}$$

Alternatively, the above solution can be written in terms of the eigenvalues of Q and the corresponding projection operators as follows

$$\begin{aligned} f(x) &= \mathbf{E}_x \left[\phi\left(S_{\tau_0}\right)\right] + \sum_{\lambda} \frac{1}{1+\lambda} \mathbf{E}_x \left[\tau_0 \Pi_\lambda (I - \widetilde{P})\phi(S_{\tau_0})\right] \\ &\quad + \mathbf{E}_x \left[\sum_{k=0}^{\tau_0 - 1} (k+1)\psi(S_k) \right] - \sum_{\lambda} \frac{1}{1+\lambda} \mathbf{E}_x \left[\tau_0 \Pi_\lambda h\left(S_{\tau_0}\right)\right], \end{aligned} \tag{3.7}$$

where $\widetilde{P} f(z) := \mathbf{E}_z[f(S_{\tau_1})]$ is the operator acting on functions defined on $\partial_1\Lambda$ and

$$h(z) = \mathbf{E}_z \left[\sum_{k=0}^{\tau_1 - 1} k\psi(S_k) \right], \quad z \in \Lambda^c.$$

Note that (3.7) is a re-writing of the solution (3.6) which is a by-product of Vanderbei's proof.

Remark 8 Without the presence of η the series describing the covariances might not be absolutely summable on every graph, as discussed in an example in [19]. However in our case, that is for regular trees where we have exponential decay of correlations for φ, one can show that η_t does not play any role and can in fact be avoided altogether. Also note that since $\{\tau_i - \tau_{i-1} : i \geq 1\}$ need not be i.i.d. the terms involving excursion times to Λ^c in (3.6) must be dealt carefully.

We also note that the representation (3.7) is not directly stated as a theorem in [19] but if one goes through the proof of Theorem 4 in [19] then it follows immediately.

3.2 Back to Regular Trees

In our set-up, V_n consists in the first n generations of the regular tree. Note that $V_n \cup \partial_1 V_n$ is finite. It follows from Lemma 11 that $\mathbf{E}_x[\tau_0^2] < \infty$ for all $x \in \mathbb{T}_m$ with $m \geq 3$, so that (i)-(ii) are satisfied. It can be easily proved using the theory of electrical networks that the simple random walk on $\mathbb{T}_m$ is transient for all $m \geq 3$. Using the solution (3.6) we have the random walk representation of $G_n(x, y)$ as follows:

$$G_n(x, y) = \lim_{t\to\infty} \mathbf{E}_x \left[\sum_{j=0}^{\eta_t} (-1)^j M_{j-1} \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1})\, \mathbb{1}_{[S_k = y]} \right]. \tag{3.8}$$

We have used $\phi(z) = 0$ and $\psi(z) = \mathbb{1}_{[z=y]}$ in equation (3.6).

We write $G_n(x, y)$ as

$$G_n(x, y) = \overline{G}_n(x, y) - E_n(x, y), \tag{3.9}$$

where

$$\overline{G}_n(x, y) := \mathbf{E}_x \left[\sum_{k=0}^{\tau_0 - 1} (k+1)\, \mathbb{1}_{[S_k = y]} \right],$$

$$E_n(x, y) := \lim_{t \to \infty} \mathbf{E}_x \left[\sum_{j=1}^{\eta_t} (-1)^{j-1} M_{j-1} \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1})\, \mathbb{1}_{[S_k = y]} \right]. \tag{3.10}$$

Note that $\overline{G}_n(x, y) = \mathbf{E}_{x,y} \left[\sum_{k=0}^{\tau_0 - 1} \sum_{\ell=0}^{\tau_0' - 1} \mathbb{1}_{[S_k = S_\ell']} \right]$ where S_k and S_ℓ' are two independent simple random walks starting from x and y respectively, and τ_0 and τ_0' are their first visit times to V_n^c respectively. $\overline{G}_n(x, y)$ plays crucial role in the study of the membrane model: in the $\mathbb{Z}^d$ case, it was shown in [13] that G_n and $\overline{G}_n$ are close in the bulk of the domain. We will also see here that $E_n(x, y)$ plays a role of the error term. We observe from (3.9) and (3.7) that

$$E_n(x, y) = \mathbf{E}_x \left[\tau_0 \sum_\lambda \frac{1}{1+\lambda} \Pi_\lambda h(S_{\tau_0}) \right], \tag{3.11}$$

where $h(z) = \mathbf{E}_z \left[\sum_{k=0}^{\tau_1 - 1} k\, \mathbb{1}_{[S_k = y]} \right]$ for $z \in V_n^c$.

4 Proof of Theorem 2

If the infinite volume limit exists then it is supposed to have the covariance function G. We first show $G(x, y) = \sum_{k=0}^{\infty} (k+1) \mathbf{P}_x(S_k = y)$ can be computed in terms of $d(x, y)$ for a m-regular tree and that it has exponential decay in the distance $d(x, y)$.

Lemma 9 *We have for any $x, y \in \mathbb{T}_m$ that*

$$G(x, y) = \frac{(d(x, y) + 1) m(m-1)(m-2) + 2(m-1)}{(m-2)^3 (m-1)^{d(x,y)}}.$$

Proof We define the Green's function of the simple random walk on $\mathbb{T}_m$ as the power series

$$\Gamma(x, y|\mathbf{z}) := \sum_{k=0}^{\infty} \mathbf{P}_x(S_k = y) \mathbf{z}^k, \quad x, y \in \mathbb{T}_m, \quad \mathbf{z} \in \mathbb{C}.$$

From **(author?)** [20, Lemma 1.24] we have

$$\Gamma(x, y|\mathbf{z}) = \frac{2(m-1)}{m - 2 + \sqrt{m^2 - 4(m-1)\mathbf{z}^2}} \left(\frac{m - \sqrt{m^2 - 4(m-1)\mathbf{z}^2}}{2(m-1)\mathbf{z}} \right)^{d(x,y)}. \tag{4.1}$$

We fix $x, y \in \mathbb{T}_m$ and write $d = d(x, y)$, $g(\mathbf{z}) := \Gamma(x, y|\mathbf{z})$ for $\mathbf{z} \in \mathbb{C}$. Now observe that

$$G(x, y) = g'(1) + g(1).$$

From (4.1) we get

$$\log(g(\mathbf{z})) = \log(2(m-1)) - \log\left(m - 2 + \sqrt{m^2 - 4(m-1)\mathbf{z}^2}\right) + d\log\left(m - \sqrt{m^2 - 4(m-1)\mathbf{z}^2}\right) - d\log(2(m-1)) - d\log\mathbf{z}.$$

So taking a derivative we have

$$\frac{g'(\mathbf{z})}{g(\mathbf{z})} = \frac{8(m-1)\mathbf{z}}{2\left(m-2+\sqrt{m^2-4(m-1)\mathbf{z}^2}\right)\sqrt{m^2-4(m-1)\mathbf{z}^2}} + \frac{8d(m-1)\mathbf{z}}{2\left(m-\sqrt{m^2-4(m-1)\mathbf{z}^2}\right)\sqrt{m^2-4(m-1)\mathbf{z}^2}} - \frac{d}{\mathbf{z}}$$

and hence evaluation at $\mathbf{z} = 1$ gives

$$\frac{g'(1)}{g(1)} = \frac{2(m-1)}{(m-2)^2} + \frac{2d(m-1)}{m-2} - d = \frac{2(m-1) + dm(m-2)}{(m-2)^2}.$$

Also

$$g(1) = \frac{1}{(m-2)(m-1)^{d-1}}. \tag{4.2}$$

Now we obtain

$$G(x, y) = g'(1) + g(1) = g(1)\left(\frac{g'(1)}{g(1)} + 1\right) = \frac{2(m-1) + dm(m-2) + (m-2)^2}{(m-2)^3(m-1)^{d-1}} = \frac{(d+1)m(m-2) + 2}{(m-2)^3(m-1)^{d-1}}.$$

□

The behavior of G depends crucially on the graph distance $d(x, y)$ between two points x and y on the tree. We would need an estimate on the number of points (x, y) which are at a fixed distance k. The following lemma gives a bound on this.

Lemma 10 *Let*

$$C_k := |\{(x, y) \in V_n \times V_n : d(x, y) = k\}|.$$

Then

$$C_k \leq C(m-1)^{n + \lfloor \frac{k}{2} \rfloor},$$

where C is a constant which depends on m.

Proof Let $e(x) := d(o, x)$ and for any $\ell > 0$ define

$$\partial_0 V_\ell := \{x \in V_n : e(x) = \ell\}.$$

In other words, $\partial_0 V_\ell$ is the set of all leaf-vertices in V_ℓ. Let us now fix any vertex in $\partial_0 V_{n-\ell}$ and count the number of y's in V_n such that $d(x, y) = k$. Notice that for any $x \in \mathbb{T}_m \setminus \{o\}$, we have $e(y) = e(x) \pm 1$ for all y with $x \sim y$. Moreover, $e(y) = e(x) - 1$ holds for only one such y and $e(y) = e(x) + 1$ holds for remaining $(m-1)$ many such y. In other words, from any $x \in \mathbb{T}_m \setminus \{o\}$ there is only one way to move closer to o and $(m-1)$ way to move farther away from o. Now let us first consider the case when $k \leq n$. If $0 \leq \ell < k$, then for any $x \in \partial_0 V_{n-\ell}$

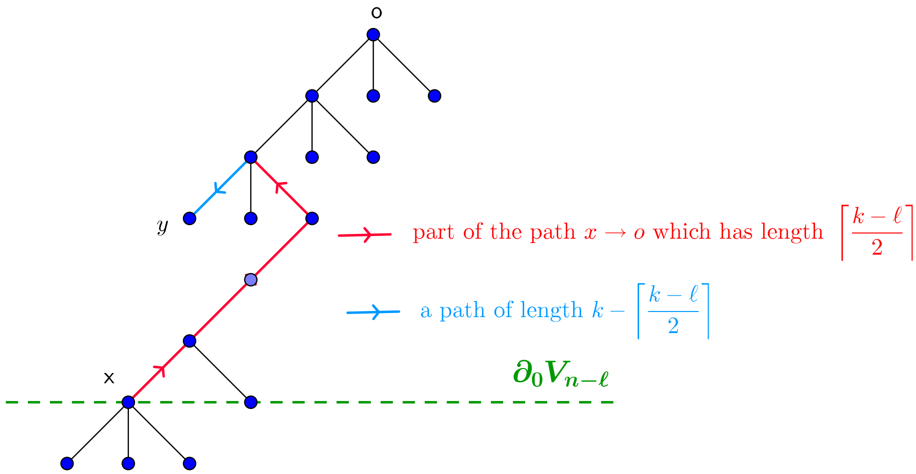

Fig. 1 Case $0 \le \ell < k \le n$ for the proof of Lemma 10. If $d(x, y) = k$, in order to reach y from x one must move to their least common ancestor in $\lceil (k-\ell)/2 \rceil$ steps and from there move to y in $k - \lceil (k-\ell)/2 \rceil$ steps. The 3−ary tree is partially drawn for simplicity

the number of vertices in V_n which are at a distance k from x is $(m-1)^{\lfloor (k-\ell)/2 \rfloor + \ell}$. This is because the unique path of length k from x to some vertex in V_n must consists of first $\lceil (k-l)/2 \rceil$ steps moving closer to o and the remaining steps moving in any of the $(m-1)$ available directions. The situation is explained graphically in Figure 1. On the other hand if $\ell \ge k$, then for any $x \in \partial_0 V_{n-\ell}$ the number of vertices in V_n which are at a distance k from x is $m(m-1)^{k-1}$. Also, note that for any $j \ge 1$ there are $m(m-1)^{j-1}$ vertices in $\partial_0 V_j$. Therefore in this case

$$\begin{aligned} C_k &= \sum_{\ell=0}^{k-1} m(m-1)^{n-\ell-1}(m-1)^{\lfloor \frac{k-\ell}{2} \rfloor + \ell} \\ &\quad + \sum_{\ell=k}^{n-1} m(m-1)^{n-\ell-1} m(m-1)^{k-1} + m(m-1)^{k-1} \\ &= m(m-1)^{n+\lfloor \frac{k}{2} \rfloor - 1} \sum_{\ell=0}^{k-1} (m-1)^{-\lfloor \frac{\ell}{2} \rfloor} + m^2(m-1)^{n-2} \sum_{\ell=0}^{n-k-1} (m-1)^{-\ell} + m(m-1)^{k-1} \\ &\le C(m-1)^{n+\lfloor \frac{k}{2} \rfloor}. \end{aligned}$$

Now we consider the case when $k > n$. In this case for any $x \in \partial_0 V_{n-\ell}$ the maximum number of vertices which are at a distance k from x is $(m-1)^{\lfloor (k-\ell)/2 \rfloor + \ell}$ (here we are over-counting and for $k > 2n - \ell$ this number is 0). Therefore

$$\begin{aligned} C_k &\le \sum_{\ell=0}^{n} m(m-1)^{n-\ell-1}(m-1)^{\lfloor \frac{k-\ell}{2} \rfloor + \ell} \\ &= m(m-1)^{n-1+\lfloor \frac{k}{2} \rfloor} \sum_{\ell=0}^{n} (m-1)^{-\lfloor \frac{\ell}{2} \rfloor} \le C(m-1)^{n+\lfloor \frac{k}{2} \rfloor}. \end{aligned}$$

□

Our final lemma before the proof of Theorem 2 gives an estimate on the second moment of the exit time.

Lemma 11 *For any* $x \in V_n$

$$\mathbf{E}_x[\tau_0^2] \le Cd(x, \partial_1 V_n)^2. \tag{4.3}$$

Proof Observe that $\mathbf{E}_x\left[\sum_{z\in V_n}\sum_{\ell=0}^{\tau_0-1}\ell\,\mathbb{1}_{[S_\ell=z]}\right] = \mathbf{E}_x\left[\tau_0(\tau_0-1)/2\right]$ which implies

$$\begin{aligned}\mathbf{E}_x\left[\tau_0^2\right] &\le 2\mathbf{E}_x\left[\sum_{z\in V_n}\sum_{\ell=0}^{\tau_0-1}\ell\,\mathbb{1}_{[S_\ell=z]}\right]\\ &\le 2\sum_{z\in V_n} G(x,z).\end{aligned}$$

By using Lemma 9 we get

$$\mathbf{E}_x\left[\tau_0^2\right] \le C\sum_{z\in V_n} d(x,z)(m-1)^{-d(x,z)} = C\sum_{k=0}^{2n}\sum_{z:d(x,z)=k} k(m-1)^{-k}.$$

Now suppose $\ell = d(0,x)$. With arguments similar to the proof of Lemma 10, we have

$$|\{z : d(x,z) = k\}| \le \begin{cases} m(m-1)^k & \text{if } 0 \le k \le n-\ell \\ (m-1)^{\frac{k}{2}+\frac{n-\ell}{2}} & \text{if } n-\ell+1 \le k \le n+\ell. \end{cases}$$

Hence we have

$$\begin{aligned}\mathbf{E}_x[\tau_0^2] &\le C\sum_{k=0}^{n-\ell} m(m-1)^k k(m-1)^{-k} + C\sum_{k=n-\ell+1}^{n+\ell}(m-1)^{\frac{k}{2}+\frac{n-\ell}{2}}k(m-1)^{-k}\\ &\le C\sum_{k=0}^{n-\ell}k + \sum_{k=1}^{2\ell}(k+n-\ell)(m-1)^{-\frac{k}{2}} \le C(n-\ell)^2 \le Cd(x,\partial_1 V_n)^2.\end{aligned}$$

□

***Proof of Theorem* 2** Since the random walk on $\mathbb{T}_m$ starting from vertex x is transient, τ_0 is finite almost surely and $\tau_0 \ge n - d(o,x)$ for all $n \ge 1$. Therefore τ_0 increases to ∞ as $n \to \infty$. Hence as an immediate conclusion $\{\overline{G}_n(x,y)\}_{n\ge1}$ is an increasing sequence. By monotone convergence theorem we have

$$\lim_{n\to\infty}\overline{G}_n(x,y) = \mathbf{E}_x\left(\sum_{k=0}^{\infty}(k+1)\,\mathbb{1}_{[S_k=y]}\right).$$

We now show that $|E_n(x,y)| \to 0$ as n tends to infinity so that we have

$$\lim_{n\to\infty} G_n(x,y) = \mathbf{E}_x\left(\sum_{k=0}^{\infty}(k+1)\,\mathbb{1}_{[S_k=y]}\right).$$

As the measures under consideration are Gaussian, this will complete the proof. Recall the representation of $E_n(x,y)$ from (3.11) with $h(z) = \mathbf{E}_z\left[\sum_{k=0}^{\tau_1-1}k\,\mathbb{1}_{[S_k=y]}\right]$. Since Q has eigenvalues λ it can be seen that $\left(\frac{1}{I+Q}\right) = \sum_\lambda \frac{1}{1+\lambda}\Pi_\lambda$. So we can rewrite (3.11) as

$$
\begin{aligned}
E_n(x,y) &= \mathbf{E}_x\left[\tau_0 \left(\frac{1}{I+Q}\right) h(S_{\tau_0})\right] \\
&= \mathbf{E}_x\left[\sum_{z\in V_n^c} \tau_0\, \mathbb{1}_{[S_{\tau_0}=z]}\left(\frac{1}{I+Q}\right) h(z)\right] = \sum_{z\in V_n^c} \mathbf{E}_x\left[\tau_0\, \mathbb{1}_{[S_{\tau_0}=z]}\right]\left(\frac{1}{I+Q}\right) h(z).
\end{aligned}
$$

By Cauchy-Schwarz inequality and the fact that $\| (I+Q)^{-1} \| \le 1$ we get

$$
\begin{aligned}
E_n(x,y)^2 &\le \sum_{z\in V_n^c}\left(\mathbf{E}_x\left[\tau_0\, \mathbb{1}_{[S_{\tau_0}=z]}\right]\right)^2 \sum_{z\in V_n^c} h(z)^2 \\
&= \sum_{z\in \partial_1 V_n}\left(\mathbf{E}_x\left[\tau_0\, \mathbb{1}_{[S_{\tau_0}=z]}\right]\right)^2 \sum_{z\in \partial_1 V_n} h(z)^2 \\
&\le \sum_{z\in \partial_1 V_n} \mathbf{E}_x[\tau_0^2]\mathbf{P}_x(S_{\tau_0}=z) \sum_{z\in \partial_1 V_n} G(z,y)^2 \\
&\le \mathbf{E}_x[\tau_0^2] \sum_{z\in \partial_1 V_n} G(z,y)^2. \qquad (4.4)
\end{aligned}
$$

Now we obtain a bound for the second factor in (4.4). We have for $y \in \partial_0 V_\ell$

$$
\begin{aligned}
\sum_{z\in \partial_1 V_n} G(z,y)^2 &\le C \sum_{k=n-\ell+1}^{n+\ell+1} \sum_{z:d(z,y)=k} k^2(m-1)^{-2k} \le C \sum_{k=n-\ell+1}^{n+\ell+1} m(m-1)^{k-1}k^2(m-1)^{-2k} \\
&\le C \sum_{k=1}^{2\ell}(k+n-\ell)^2(m-1)^{-(k+n-\ell)} \le C(n-\ell)^2(m-1)^{-(n-\ell)}.
\end{aligned}
$$

Thus we have

$$
\sum_{z\in \partial_1 V_n} G(z,y)^2 \le Cd(y,\partial_1 V_n)^2(m-1)^{-d(y,\partial_1 V_n)}. \qquad (4.5)
$$

Plugging in the bounds (4.3), (4.5) in (4.4) we obtain

$$
E_n(x,y)^2 \le Cd(x,\partial_1 V_n)^2 d(y,\partial_1 V_n)^2 (m-1)^{-d(y,\partial_1 V_n)}.
$$

From symmetry we can conclude

$$
E_n(x,y)^2 \le Cd(x,\partial_1 V_n)^2 d(y,\partial_1 V_n)^2 \min\{(m-1)^{-d(x,\partial_1 V_n)}, (m-1)^{-d(y,\partial_1 V_n)}\}. \qquad (4.6)
$$

It follows from (4.6) that $|E_n(x,y)| \to 0$ as $n \to \infty$. □

An alternative proof of (4.6) using the maximum principle for harmonic functions is provided in Appendix A.

5 Proof of Theorem 3

In this section we prove Theorem 3.

***Proof of Theorem* 3** For any $x \in \mathbb{T}_m$, we define $\psi_x := \varphi_x / \sqrt{G(o,o)}$. Then for each x, ψ_x is a standard Gaussian random variable. Recall that for any n

$$B_n = \frac{b_n}{\sqrt{G(o,o)}} = \sqrt{2\log N} - \frac{\log\log N + \log(4\pi)}{2\sqrt{2\log N}}, \quad A_n = B_n^{-1}.$$

We set

$$u_n(\theta) := A_n\theta + B_n, \ \ \theta \in \mathbb{R}.$$

Let $\theta \in \mathbb{R}$ be fixed and define

$$W_n := \sum_{x\in V_n} \mathbb{1}_{[\psi_x > u_n(\theta)]}, \ \ \lambda_n := \mathbf{E}[W_n] = \sum_{x\in V_n} \mathbf{P}(\psi_x > u_n(\theta)).$$

Let $Poi(\lambda)$ denote a Poisson random variable with parameter λ. We shall use a Binomial-to-Poisson approximation by [12]. By Theorem 3.1 of [12] we get

$$d_{TV}(W_n, Poi(\lambda_n)) \le \frac{1-\mathrm{e}^{-\lambda_n}}{\lambda_n}\left[\sum_{x\in V_n} \mathbf{P}(\psi_x > u_n(\theta))^2 \right. \\ \left. + \sum_{x,y\in V_n, x\ne y} \left|\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)\right| \right], \tag{5.1}$$

where d_{TV} is the total variation distance. We want to prove that $d_{TV}(W_n, Poi(\lambda_n))$ goes to zero as n tends to infinity. Once we prove it, we will have

$$|\mathbf{P}(W_n = 0) - \mathrm{e}^{-\lambda_n}| \to 0.$$

But

$$\mathbf{P}(W_n = 0) = P\left(\max_{x\in V_n} \psi_x \le u_n(\theta)\right) = P\left(\frac{\max_{x\in V_n}\varphi_x - b_n}{a_n} \le \theta\right).$$

Using Mill's ratio

$$\left(1 - \frac{1}{t^2}\right)\frac{\mathrm{e}^{-\frac{t^2}{2}}}{\sqrt{2\pi}t} \le P(\mathcal{N}(0,1) > t) \le \frac{\mathrm{e}^{-\frac{t^2}{2}}}{\sqrt{2\pi}t} \qquad t > 0,$$

one can show that $\lambda_n = N\mathbf{P}(\psi_o > u_n(\theta))$ converges to $\mathrm{e}^{-\theta}$ as n tends to infinity. Hence the proof will be complete.

We now obtain bounds for the terms in (5.1) to prove that $d_{TV}(W_n, Poi(\lambda_n))$ goes to zero as n tends to infinity.

Another use of Mill's ratio gives

$$\frac{1-\mathrm{e}^{-\lambda_n}}{\lambda_n}\sum_{x\in V_n}\mathbf{P}(\psi_x > u_n(\theta))^2 \to 0.$$

Now we give a bound on the other term in (5.1). Let $x, y \in V_n$ with $d(x,y) = k \ge 1$ and define $r_k := \mathbf{Cov}(\psi_x, \psi_y)$. From Lemma 9 it follows that r_k depends only on k, not on x or y, and moreover $0 < r_k < 1$ for all $k \ge 1$. Now from Lemma 3.4 of [12] we obtain the following bounds:

$$0 \le \mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right) \le CN^{-\frac{2}{1+r_k}} \tag{5.2}$$

$$0 \le \mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right) \le C r_k N^{-2} \log N \, \mathrm{e}^{2r_k \log N}. \tag{5.3}$$

We fix δ such that $0 \le \delta < (1 - r_1)/(1 + r_1)$. We have

$$\begin{aligned} I_n := & \frac{1 - \mathrm{e}^{-\lambda_n}}{\lambda_n} \sum_{x, y \in V_n, x \neq y} \left|\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)\right| \\ & \le C \sum_{k=1}^{2n} \sum_{x, y \in V_n, \, d(x,y) = k} \mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right). \end{aligned}$$

Now we use the bounds (5.2), (5.3) and get the following bound for the above term:

$$\begin{aligned} I_n & \le C \sum_{k=1}^{\lfloor 2n\delta \rfloor} C_k N^{-\frac{2}{1+r_k}} + C \sum_{\lfloor 2n\delta \rfloor + 1}^{2n} C_k N^{-2} \log N \mathrm{e}^{2r_k \log N} \\ & \le C \sum_{k=1}^{\lfloor 2n\delta \rfloor} (m-1)^{n + \lfloor \frac{k}{2} \rfloor} N^{-\frac{2}{1+r_k}} + C \sum_{k = \lfloor 2n\delta \rfloor + 1}^{2n} (m-1)^{n + \lfloor \frac{k}{2} \rfloor} k (m-1)^{-k} N^{-2} \log N \mathrm{e}^{2 r_{\lfloor 2n\delta \rfloor} \log N} \\ & \le C (m-1)^{n + n\delta - \frac{2n}{1+r_1}} + C n^3 (m-1)^{-n + 2n r_{\lfloor 2n\delta \rfloor}}. \end{aligned}$$

Note that in the second inequality we have used Lemma 10 and the fact that r_k is decreasing in k with $r_k \le Ck(m-1)^{-k}$, which follows from Lemma 9. Now observe that by definition $1 + \delta < 2/(1 + r_1)$ and $r_{\lfloor 2n\delta \rfloor} < 1/2$ for large enough n. Thus I_n goes to 0 as n tends to infinity and we conclude $d_{TV}(W_n, Poi(\lambda_n))$ goes to zero as n tends to infinity. □

6 Proof of Theorem 4

In this section we prove Theorem 4. We use the following

Lemma 12 *For any $x \in V_n$ one has $G_n(x, x) \le G(o, o)$.*

Proof The proof can be readily adapted from that of **(author?)** [4, Corollary 3.2] which is carried out for $\mathbb{Z}^d$. □

The following lemma gives a bound on $\mathbf{E}_x[\tau_0]$ in terms of the distance of the point x from the boundary.

Lemma 13 *For $x \in V_n$*

$$\mathbf{E}_x[\tau_0] \le C_1(m) d(x, \partial_1 V_n), \tag{6.1}$$

where

$$C_1(m) = \frac{(m-1)^{\frac{3}{2}}}{(m-2)(\sqrt{m-1} - 1)} + \frac{m}{m-2}. \tag{6.2}$$

Proof Using (4.2) we have

$$\begin{aligned} \mathbf{E}_x[\tau_0] = \mathbf{E}_x \left[\sum_{y \in V_n} \sum_{\ell=0}^{\tau_0 - 1} \mathbb{1}_{[S_\ell = y]} \right] \le \sum_{y \in V_n} \mathbf{E}_x \left[\sum_{\ell=0}^{\infty} \mathbb{1}_{[S_\ell = y]} \right] \\ = \sum_{y \in V_n} \frac{1}{(m-2)(m-1)^{d(x,y) - 1}}. \end{aligned}$$

Suppose $d(o,x)=k$. Then similarly to the proof of Lemma 10 we argue that

$$|\{y\in V_n : d(x,y)=\ell\}| = \begin{cases} m(m-1)^{\ell-1}, & \text{if } 1\le \ell \le n-k \\ (m-1)^{\lfloor \frac{l-n+k}{2}\rfloor+(n-k)}, & \text{if } n-k<\ell\le n+k \\ 0, & \text{if } \ell > n+k \end{cases}.$$

Therefore splitting the sum according to the distance $d(x,y)$ we get,

$$\begin{aligned} \mathbf{E}_x[\tau_0] &= \frac{m-1}{m-2}\sum_{y\in V_n}(m-1)^{-d(x,y)} \\ &\le \frac{m-1}{m-2}\left[1+\sum_{\ell=1}^{n-k}\sum_{y\in V_n : d(x,y)=\ell}(m-1)^{-\ell}+\sum_{\ell=n-k+1}^{n+k}\sum_{y\in V_n : d(x,y)=\ell}(m-1)^{-\ell}\right] \\ &= \frac{m-1}{m-2}\left[1+\sum_{\ell=1}^{n-k} m(m-1)^{\ell-1}(m-1)^{-\ell}+\sum_{\ell=n-k+1}^{n+k}(m-1)^{\lfloor\frac{\ell-n+k}{2}\rfloor+(n-k)}(m-1)^{-\ell}\right]. \end{aligned}$$

As $\lfloor\frac{\ell-n+k}{2}\rfloor \le \frac{\ell}{2}-\frac{n-k}{2}$, we have

$$\begin{aligned} \mathbf{E}_x[\tau_0] &\le \frac{m-1}{m-2}\left[1+\frac{m}{m-1}(n-k)+\sum_{\ell=n-k+1}^{n+k}(m-1)^{\frac{\ell}{2}-\frac{n-k}{2}+n-k-\ell}\right] \\ &= \frac{m-1}{m-2}\left[1+\frac{m}{m-1}(n-k)+\sum_{\ell=1}^{2k}(m-1)^{-\frac{\ell}{2}}\right] \\ &= \frac{m-1}{m-2}\left[1+\frac{m}{m-1}(n-k)+(m-1)^{-\frac12}\frac{1-(m-1)^{-k}}{1-(m-1)^{-\frac12}}\right] \\ &\le \left(\frac{(m-1)^{\frac32}}{(m-2)(\sqrt{m-1}-1)}+\frac{m}{m-2}\right)d(x,\partial_1 V_n). \end{aligned}$$

□

The following bound will also be useful.

Lemma 14 *For any $z\in\partial_1 V_n$*

$$\mathbf{E}_z\left[\sum_{k=0}^{\tau_1-1} k\,\mathbb{1}_{[S_k=y]}\right] \le C_2(m)d(y,\partial_1 V_n)(m-1)^{-d(y,\partial_1 V_n)},$$

where

$$C_2(m)=\left(\frac{2(m-1)^2}{(m-2)^3}+\frac{m(m-1)}{(m-2)^2}\right). \tag{6.3}$$

Proof We have

$$\begin{aligned} \mathbf{E}_z\left[\sum_{k=0}^{\tau_1-1} k\,\mathbb{1}_{[S_k=y]}\right] &\le \mathbf{E}_z\left[\sum_{k=0}^{\infty} k\,\mathbb{1}_{[S_k=y]}\right] \\ &= \frac{2(m-1)+d(z,y)m(m-2)}{(m-2)^3(m-1)^{d(z,y)-1}} \end{aligned}$$

$$\leq \left(\frac{2(m-1)^2}{(m-2)^3} + \frac{m(m-1)}{(m-2)^2} \right) d(y, \partial_1 V_n)(m-1)^{-d(y,\partial_1 V_n)}.$$

□

We now proceed to prove Theorem 4.

***Proof of Theorem* 4** First we prove an upper bound for the expected maximum using a standard trick. Using Jensen's inequality and Lemma 12 we have for any $\beta > 0$

$$\mathbf{E}_n \left[\max_{x \in V_n} \varphi_x \right] \leq \frac{1}{\beta} \log \left(\mathbf{E}_n \left[\sum_{x \in V_n} \mathrm{e}^{\beta \varphi_x} \right] \right) = \frac{1}{\beta} \log \left(\sum_{x \in V_n} \mathrm{e}^{\frac{\beta^2}{2} G_n(x,x)} \right)$$
$$\leq \frac{1}{\beta} \log \left(\sum_{x \in V_n} \mathrm{e}^{\frac{\beta^2}{2} G(o,o)} \right) = \frac{\beta}{2} G(o,o) + \frac{1}{\beta} \log N.$$

Optimizing over β we obtain

$$\mathbf{E}_n \left[\max_{x \in V_n} \varphi_x \right] \leq \sqrt{2G(o,o) \log N}.$$

Thus

$$\limsup_{n \to \infty} \frac{\mathbf{E}_n \left[\max_{x \in V_n} \varphi_x \right]}{\sqrt{2 \log N}} \leq \sqrt{G(o,o)}. \tag{6.4}$$

Next we prove the lower bound for the limes inferior. We use a Gaussian comparison inequality on an appropriate set of vertices. For this we need a lower bound on $G_n(x,x)$ for x in an appropriate subset of V_n. Using (3.9) we write

$$G_n(x,y) = G(x,y) - \overline{E}_n(x,y) - E_n(x,y), \tag{6.5}$$

where

$$\overline{E}_n(x,y) := \mathbf{E}_x \left[\sum_{k=\tau_0}^{\infty} (k+1) \mathbb{1}_{[S_k = y]} \right].$$

Our target is to obtain a bound for $\overline{E}_n(x,y)$. We have using Lemma 13, Lemma 14 and (4.1) with $\mathbf{z} = 1$

$$\overline{E}_n(x,y) = \mathbf{E}_x \left[\mathbf{E}_{S_{\tau_0}} \left[\sum_{k=0}^{\infty} k \, \mathbb{1}_{[S_k = y]} \right] \right] + \mathbf{E}_x \left[(\tau_0 + 1) \mathbf{E}_{S_{\tau_0}} \left[\sum_{k=0}^{\infty} \mathbb{1}_{[S_k = y]} \right] \right]$$
$$\leq C_2(m) d(y, \partial_1 V_n)(m-1)^{-d(y,\partial_1 V_n)} + \frac{m-1}{m-2} (m-1)^{-d(y,\partial_1 V_n)} \mathbf{E}_x[\tau_0 + 1]$$
$$\leq C_2(m) d(y, \partial_1 V_n)(m-1)^{-d(y,\partial_1 V_n)} + \frac{m-1}{m-2}$$
$$\times (1 + C_1(m) d(x, \partial_1 V_n)) (m-1)^{-d(y,\partial_1 V_n)}. \tag{6.6}$$

To prove the bound for the limes inferior we define a subset U_n of V_n as follows. For each $z \in \partial_0 V_{n-2\lfloor \log n \rfloor} = \{x \in V_n : d(o,x) = n - 2\lfloor \log n \rfloor\}$, choose exactly one $y_z \in \partial_0 V_{n-\lfloor \log n \rfloor}$ such that $d(z, y_z) = \lfloor \log n \rfloor$. Then define

$$U_n := \{y_z : z \in \partial_0 V_{n-2\lfloor \log n \rfloor}\}.$$

Note that $U_n \subset \partial_0 V_{n-\lfloor \log n \rfloor}$ but $|U_n| = |\partial_0 V_{n-2\lfloor \log n \rfloor}| = m(m-1)^{n-2\lfloor \log n \rfloor - 1}$. Also from the definition of U_n it follows that for any $x, y \in U_n, d(x, y) \geq 2\lfloor \log n \rfloor$ and for any $x \in U_n$, $d(x, \partial_1 V_n) = \lfloor \log n \rfloor + 1$. Now using the crude bound for $E_n(x, y)$ in (4.6) we have that for any $x, y \in U_n$

$$E_n(x, y) \leq C_0(m)(\log n)^2(m-1)^{-\log n},$$

where $C_0(m)$ is a constant dependent on m only. Also from (6.6) we have for $x \in U_n$

$$\begin{aligned}\overline{E}_n(x, x) &\leq C_2(m) \log n (m-1)^{-\log n} + \frac{m-1}{m-2}\left(1 + C_1(m) \log n\right)(m-1)^{-\log n} \\ &\leq \frac{m-1}{m-2}\left(1 + C_1(m) + C_2(m)\right) \log n (m-1)^{-\log n}.\end{aligned}$$

Using these bounds we get from (6.5) that for any $y \in U_n$

$$\begin{aligned}G_n(y, y) \geq G(o, o) &- \frac{m-1}{m-2}\left(1 + C_1(m) + C_2(m)\right) \log n (m-1)^{-\log n} \\ &- C_0(m)(\log n)^2(m-1)^{-\log n}.\end{aligned} \tag{6.7}$$

Also from Lemma 9 we have for $y, \ y' \in U_n$

$$G(y, y') \leq C_3(m) \log n (m-1)^{-2\log n}$$

and hence

$$\begin{aligned}G_n(y, y') &\leq G(y, y') + |E_n(y, y')| \\ &\leq C_3(m) \log n (m-1)^{-2\log n} + C_0(m)(\log n)^2(m-1)^{-\log n}.\end{aligned} \tag{6.8}$$

Now using (6.7) and (6.8) we have for $y, y' \in U_n$

$$\begin{aligned}\mathbf{E}_n\left[(\varphi_y - \varphi_{y'})^2\right] &= G_n(y, y) + G_n(y', y') - 2G_n(y, y') \\ &\geq 2\Bigg[G(o, o) - \frac{m-1}{m-2}\left(1 + C_1(m) + C_2(m)\right) \log n (m-1)^{-\log n} \\ &\quad - C_3(m) \log n (m-1)^{-2\log n} - C_0(m)(\log n)^2(m-1)^{-\log n}\Bigg].\end{aligned} \tag{6.9}$$

We define

$$\begin{aligned}\gamma(n, m) := \Bigg[&G(o, o) - \frac{m-1}{m-2}\left(1 + C_1(m) + C_2(m)\right) \log n (m-1)^{-\log n} \\ &- C_3(m) \log n (m-1)^{-2\log n} - C_0(m)(\log n)^2(m-1)^{-\log n}\Bigg].\end{aligned}$$

Note that $\gamma(n, m) \to G(o, o)$ as $n \to \infty$. Suppose n is large enough so that $\gamma(n, m) > 0$. Let $(\xi_x)_{x \in U_n}$ be i.i.d. centered Gaussian random variables with variance $\gamma(n, m)$. Then from (6.9) we have

$$\mathbf{E}_n\left[(\varphi_x - \varphi_y)^2\right] \geq \mathbf{E}\left[(\xi_x - \xi_y)^2\right].$$

Therefore by the Sudakov-Fernique inequality [1, Theorem 2. 2. 3] we have

$$\mathbf{E}_n\left[\max_{x \in U_n} \varphi_x\right] \geq \mathbf{E}\left[\max_{x \in U_n} \xi_x\right].$$

As $U_n \subset V_n$, we get

$$\liminf_{n\to\infty} \frac{\mathbf{E}_n\left[\max_{x\in V_n}\varphi_x\right]}{\sqrt{2\log N}} \geq \liminf_{n\to\infty} \frac{\mathbf{E}_n\left[\max_{x\in U_n}\xi_x\right]}{\sqrt{2\log|U_n|}}\sqrt{\frac{\log|U_n|}{\log N}}.$$

But

$$\frac{\log|U_n|}{\log N} = \frac{n-2\log n}{n} \overset{n\to\infty}{\to} 1.$$

Therefore

$$\liminf_{n\to\infty} \frac{\mathbf{E}_n\left[\max_{x\in V_n}\varphi_x\right]}{\sqrt{2\log N}} \geq \sqrt{G(o,o)}. \tag{6.10}$$

So the result follows now combining the lower bound (6.10) with the upper bound (6.4). □

7 Proof of Theorem 5

In this section we prove Theorem 5. Before proving, we state two estimates which are crucially used in the proof. Recall the crucial error term in the Vanderbei's representation (3.10)

$$E_n(x,y) = \lim_{t\to\infty}\mathbf{E}_x\left[\sum_{j=1}^{\eta_t}(-1)^{j-1}M_{j-1}\sum_{k=\tau_{j-1}}^{\tau_j-1}(k-\tau_{j-1})\,\mathbb{1}_{[S_k=y]}\right].$$

Previously in (4.6) we showed that

$$E_n(x,y) \leq Cd(x,\partial_1V_n)d(y,\partial_1V_n)\min\{(m-1)^{-\frac{1}{2}d(x,\partial_1V_n)},(m-1)^{-\frac{1}{2}d(y,\partial_1V_n)}\}.$$

This bound does not say anything about the dependency of the error on $d(x,y)$. We improve the bound to get a dependence on the distance between the two points and this is crucial for our proof.

Lemma 15 *Let $x,y\in V_n$. For $m\geq 5$ and any $0\leq J_0<\infty$ we have*

$$\begin{aligned}|E_n(x,y)| \leq\ & J_0(2J_0+1)^{4J_0+2}3^{4J_0^2}\left(\frac{4(m-1)}{m-2}\right)^{2J_0+1}\frac{m-1}{m-2}d^{2J_0+1}(m-1)^{-d(x,y)}\\ &+\frac{C_1(m)C_2(m)}{\left(1-\frac{C_1(m)}{m}\right)}d(x,\partial_1V_n)d(y,\partial_1V_n)(m-1)^{-\max\{d(x,\partial_1V_n),d(y,\partial_1V_n)\}}\left(\frac{C_1(m)}{m}\right)^{J_0},\end{aligned} \tag{7.1}$$

where $C_1(m)$ and $C_2(m)$ are constants defined in (6.2) and (6.3) respectively.

As the proof requires some lengthy computations we devote Sect. 8 to it. Note that when we take $J_0=0$, the above bound improves (4.6) and we understand the constants better as well. In the proof of Theorem 5 we shall use $J_0=0$ and $J_0=\log(d(x,y))$.

We know that $G_n(x,x)\to G(o,o)$ but we do not know if this convergence is uniform as the error term depends on the distance of x from the boundary. However we see that for $m\geq 10$ we can bound $G_n(x,x)$ uniformly from below.

Lemma 16 *For $m \geq 10$ there is a positive constant $C_3(m)$ which converges to 1 as $m \to \infty$ such that*

$$\inf_{x \in V_n} G_n(x, x) \geq C_3(m). \tag{7.2}$$

Proof Note that $\overline{G}_n(x, x) \geq 1$. Taking $J_0 = 0$ in (7.1) we get for $m \geq 5$

$$|E_n(x, x)| \leq \frac{C_1(m)C_2(m)}{\left(1 - \frac{C_1(m)}{m}\right)} d(x, \partial_1 V_n)^2 (m-1)^{-d(x, \partial_1 V_n)} \leq \frac{C_1(m)C_2(m)}{(m-1)\left(1 - \frac{C_1(m)}{m}\right)}.$$

Therefore

$$G_n(x, x) \geq 1 - \frac{C_1(m)C_2(m)}{(m-1)\left(1 - \frac{C_1(m)}{m}\right)} =: C_3(m).$$

But by definition of $C_1(m)$ and $C_2(m)$ in (6.2) and (6.3) respectively, it follows

$$\lim_{m \to \infty} \frac{C_1(m)C_2(m)}{(m-1)\left(1 - \frac{C_1(m)}{m}\right)} = 0.$$

One can observe that the function $m \mapsto C_3(m)$ is an increasing function for $m \geq 5$ and for $m = 9$ and 10 we compute that $C_3(9) = -0.06$ and $C_3(10) = 0.2$. Hence for $m \geq 10$ we have $G_n(x, x) \geq C_3(m) > 0$. □

We now prove Theorem 5. In the proof we again use the comparison theorem of [12].

Proof of Theorem 5 We set

$$u_n(\theta) := A_n \theta + B_n, \ \ \theta \in \mathbb{R}.$$

From the proof of Theorem 3 we observe that in this case it suffices to prove

$$\sum_{x, y \in V_n, x \neq y} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \to 0 \tag{7.3}$$

for all θ.

We define $R_n(x, y) := \mathbf{E}_n[\psi_x \psi_y]$. First we obtain a bound for $|R_n(x, y)|$. By using Lemma 9, Lemma 15 and (7.2) we obtain

$$\begin{aligned}
|R_n(x, y)| &= \frac{|G_n(x, y)|}{\sqrt{G_n(x, x) G_n(y, y)}} \\
&\leq \frac{1}{C_3(m)} \left[G(x, y) + |E_n(x, y)|\right] \\
&\leq \frac{1}{C_3(m)} \Bigg[\frac{(d(x, y) + 1)m(m-1)(m-2) - 2(m-1)}{(m-2)^3 (m-1)^{d(x,y)}} \\
&+ J_0(2J_0 + 1)^{4J_0 + 2} 3^{4J_0^2} \left(\frac{4(m-1)}{m-2}\right)^{2J_0 + 1} \frac{m-1}{m-2} d(x, y)^{2J_0 + 1} (m-1)^{-d(x,y)} \\
&+ \frac{C_1(m)C_2(m)}{\left(1 - \frac{C_1(m)}{m}\right)} d(x, \partial_1 V_n) d(y, \partial_1 V_n)(m-1)^{-\max\{d(x, \partial_1 V_n), d(y, \partial_1 V_n)\}} \left(\frac{C_1(m)}{m}\right)^{J_0} \Bigg],
\end{aligned} \tag{7.4}$$

where $0 \le J_0 < \infty$. Taking $J_0 = 0$ in (7.4) we observe that for all distinct $x, y \in V_n$

$$|R_n(x, y)| \le \frac{1}{C_3(m)} \left[\frac{2m(m-1)(m-2) - 2(m-1)}{(m-2)^3(m-1)} + \frac{C_1(m)C_2(m)}{(m-1)\left(1 - \frac{C_1(m)}{m}\right)} \right]. \quad (7.5)$$

It is easy to check that the function

$$m \mapsto \frac{1}{C_3(m)} \left[\frac{2m(m-1)(m-2) - 2(m-1)}{(m-2)^3(m-1)} + \frac{C_1(m)C_2(m)}{(m-1)\left(1 - \frac{C_1(m)}{m}\right)} \right]$$

is decreasing for $m \ge 10$. For $m = 13$ and 14 we evaluate the above expression as 1.13 and 0.89 respectively. Therefore we conclude that for all distinct $x, y \in V_n$ and for $m \ge 14$

$$|R_n(x, y)| \le \eta \quad (7.6)$$

for some fixed η with $0 < \eta < 1$. We are now ready to prove (7.3). Let $\theta \in \mathbb{R}$ be fixed. We will use the following bounds which are obtained from Lemma 3.4 of [12].

Lemma 17 *For $x, y \in V_n$ the following hold.*

(1) If $0 \le R_n(x, y) < 1$,

$$0 \le \mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right) \le CN^{-\frac{2}{1+R_n(x,y)}}. \quad (7.7)$$

(2) If $0 \le R_n(x, y) \le 1$,

$$0 \le \mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right) \le CR_n(x, y)N^{-2}\log N e^{2R_n(x,y)\log N}. \quad (7.8)$$

(3) If $-1 \le R_n(x, y) < 0$,

$$0 \ge \mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right) \ge -CN^{-2}. \quad (7.9)$$

(4) If $-1 \le R_n(x, y) \le 0$,

$$0 \ge \mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right) \ge -C|R_n(x, y)|N^{-2}\log N. \quad (7.10)$$

We write

$$\begin{aligned}
&\sum_{x,y \in V_n, x \ne y} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \\
&\quad = \sum_{x,y \in V_n, x \ne y} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \, \mathbb{1}_{[0 \le R_n(x,y) \le 1]} \\
&\quad + \sum_{x,y \in V_n, x \ne y} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \, \mathbb{1}_{[-1 \le R_n(x,y) < 0]} =: T_1 + T_2.
\end{aligned}$$

We show that both T_1 and T_2 go to zero as n tends to infinity. First we consider T_1. Let us choose ε such that $0 < \varepsilon < (1-\eta)/(1+\eta) < 1$, where η is the same as in (7.6). We now split T_1 as

$$\begin{aligned}
T_1 &= \sum_{k=1}^{2n} \sum_{x,y \in V_n, d(x,y)=k} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \, \mathbb{1}_{[0 \le R_n(x,y) \le 1]} \\
&= \sum_{k=1}^{\lfloor n\varepsilon \rfloor} \sum_{x,y \in V_n, d(x,y)=k} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \, \mathbb{1}_{[0 \le R_n(x,y) \le 1]}
\end{aligned}$$

$$+\sum_{k=\lfloor n\varepsilon\rfloor+1}^{2n}\sum_{x,y\in V_n,d(x,y)=k}\left|\mathbf{Cov}\left(\mathbb{1}_{[\psi_x>u_n(\theta)]},\mathbb{1}_{[\psi_y>u_n(\theta)]}\right)\right|\mathbb{1}_{[0\le R_n(x,y)\le 1]}. \quad (7.11)$$

Using bound (7.7) with (7.6) and Lemma 10 we observe that the first term of (7.11) is bounded by

$$C\sum_{k=1}^{\lfloor n\varepsilon\rfloor}(m-1)^{n+\lfloor\frac{k}{2}\rfloor}N^{-\frac{2}{1+\eta}}\le C(m-1)^{n+\lfloor\frac{n\varepsilon}{2}\rfloor-\frac{2n}{1+\eta}}$$

which goes to zero as n tends to infinity by the choice of ε.

For the second term in (7.11) we use the bound (7.8) together with the bound (7.4) with $J_0=\log(d(x,y))$. We get that the second term is bounded by

$$\sum_{k=\lfloor n\varepsilon\rfloor+1}^{2n}\sum_{x,y\in V_n,d(x,y)=k}CR_n(x,y)N^{-2}\log N\mathrm{e}^{2R_n(x,y)\log N}$$
$$\le C\sum_{k=\lfloor n\varepsilon\rfloor+1}^{2n}(m-1)^{n+\lfloor\frac{k}{2}\rfloor}N^{-2}\log N\left(C\frac{k}{(m-1)^k}+B_k\right)\exp\left[2\log N\left(C\frac{k}{(m-1)^k}+B_k\right)\right],$$

where

$$B_k:=\log k(2\log k+1)^{4\log k+2}3^{4(\log k)^2}\left(\frac{4(m-1)}{m-2}\right)^{2\log k+1}\frac{m-1}{m-2}k^{2\log k+1}(m-1)^{-k}$$
$$+C\left(\frac{C_1(m)}{m}\right)^{\log k}.$$

We now use the following fact for B_k whose proof is given in the end of this section.

Claim 18 For large n and for all $k\ge\lfloor n\varepsilon\rfloor$

$$B_k\le B_{\lfloor n\varepsilon\rfloor}.$$

Using the above claim we get that the second term of (7.11) is bounded by

$$Cn\left(\frac{\lfloor n\varepsilon\rfloor}{(m-1)^{\lfloor n\varepsilon\rfloor}}+B_{\lfloor n\varepsilon\rfloor}\right)\exp\left[Cn\left(\frac{\lfloor n\varepsilon\rfloor}{(m-1)^{\lfloor n\varepsilon\rfloor}}+B_{\lfloor n\varepsilon\rfloor}\right)\right].$$

Note that to show that the above bound goes to zero as n tends to infinity it is enough to prove $nB_{\lfloor n\varepsilon\rfloor}\to 0$ as $n\to\infty$. We have

$$\begin{aligned}nB_{\lfloor n\varepsilon\rfloor}&=\exp\big[\log n+\log\log\lfloor n\varepsilon\rfloor+(4\log\lfloor n\varepsilon\rfloor+2)\log(2\log\lfloor n\varepsilon\rfloor+1)\\&\quad+(4(\log\lfloor n\varepsilon\rfloor)^2)\log 3\\&\quad+(2\log\lfloor n\varepsilon\rfloor+1)\log\left(\frac{4(m-1)}{m-2}\right)+\log(\frac{m-1}{m-2})+(2\log\lfloor n\varepsilon\rfloor+1)\log\lfloor n\varepsilon\rfloor\\&\quad-\lfloor n\varepsilon\rfloor\log(m-1)\big]\\&\quad+\exp\left[\log n+\log C+\log\lfloor n\varepsilon\rfloor\log\left(\frac{C_1(m)}{m}\right)\right]\overset{n\to\infty}{\to}0.\end{aligned}$$

Here the magnitude of m is used to get that

$$\exp\left[\log n+\log C+\log\lfloor n\varepsilon\rfloor\log\left(\frac{C_1(m)}{m}\right)\right]\overset{n\to\infty}{\to}0. \quad (7.12)$$

We observe that the function $m \mapsto \log(C_1(m)/m)$ is a decreasing function and moreover that $\log(C_1(9)/9) = -1.08$. Hence (7.12) holds for all $m \geq 9$. Thus we proved that $T_1 \to 0$ as $n \to \infty$.

Next we consider T_2. For T_2 we use the bounds (7.9), (7.10) together with the bound (7.4) with $J_0 = \log(d(x, y))$ to get

$$\begin{aligned} T_2 &= \sum_{k=1}^{2n} \sum_{x,y \in V_n, d(x,y)=k} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \, \mathbb{1}_{[-1 \leq R_n(x,y) < 0]} \\ &= \sum_{k=1}^{n-1} \sum_{x,y \in V_n, d(x,y)=k} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \, \mathbb{1}_{[-1 \leq R_n(x,y) < 0]} \\ &\quad + \sum_{k=n}^{2n} \sum_{x,y \in V_n, d(x,y)=k} |\mathbf{Cov}\left(\mathbb{1}_{[\psi_x > u_n(\theta)]}, \mathbb{1}_{[\psi_y > u_n(\theta)]}\right)| \, \mathbb{1}_{[-1 \leq R_n(x,y) < 0]} \\ &\leq C \sum_{k=1}^{n-1} (m-1)^{n + \lfloor \frac{k}{2} \rfloor - 2n} + C \sum_{k=n}^{2n} (m-1)^{n + \lfloor \frac{k}{2} \rfloor - 2n} n \left(C \frac{k}{(m-1)^k} + B_k \right). \end{aligned}$$

Clearly, the first part in the above bound goes to zero as n tends to infinity. For the second part we use the fact that $B_k \leq B_n$ for all $k \geq n$ for large n which can be proved similarly as the Claim 18. Then we get that the second part is bounded by $Cn(\frac{n}{(m-1)^n} + B_n)$ which can be shown to go to zero as $n \to \infty$ similarly as in the case of T_1. Thus $T_2 \to 0$ as $n \to \infty$. This completes the proof of (7.3). □

We now prove Claim 18.

***Proof of Claim* 18** We define for $t \geq 0$

$$\begin{aligned} F(t) :=& \log t (2\log t + 1)^{4\log t + 2} 3^{4(\log t)^2} \left(\frac{4(m-1)}{m-2} \right)^{2\log t + 1} \frac{m-1}{m-2} t^{2\log t + 1} (m-1)^{-t} \\ &+ C \left(\frac{C_1(m)}{m} \right)^{\log t} \\ =& \exp\Bigg[\log\log t + (4\log t + 2)\log(2\log t + 1) + (4(\log t)^2)\log 3 \\ &+ (2\log t + 1)\log\left(\frac{4(m-1)}{m-2}\right) + \log(\frac{m-1}{m-2}) + (2\log t + 1)\log t - t\log(m-1) \Bigg] \\ &+ \exp\left[\log C + \log t \log\left(\frac{C_1(m)}{m} \right) \right]. \end{aligned}$$

Then

$$\begin{aligned} F'(t) =& \exp\Bigg[\log\log t + (4\log t + 2)\log(2\log t + 1) + (4(\log t)^2)\log 3 \\ &+ (2\log t + 1)\log\left(\frac{4(m-1)}{m-2}\right) + \log(\frac{m-1}{m-2}) + (2\log t + 1)\log t - t\log(m-1) \Bigg] \\ &\Bigg[\frac{1}{t\log t} + \frac{4}{t}\log(2\log t + 1) + \frac{2(4\log t + 2)}{t(2\log t + 1)} + \frac{8\log 3 \log t}{t} + \frac{2}{t}\log\left(\frac{4(m-1)}{m-2}\right) \\ &+ \frac{2\log t}{t} + \frac{2\log t + 1}{t} - \log(m-1) \Bigg] \\ &+ \exp\left[\log C + \log t \log\left(\frac{C_1(m)}{m} \right) \right] \left[\frac{1}{t} \log\left(\frac{C_1(m)}{m} \right) \right]. \end{aligned}$$

This shows that $F'(t) < 0$ for all $t \geq t_0$ for some t_0. Hence F is decreasing on $[t_0, \infty)$. Therefore $F(k) = B_k \leq F(\lfloor n\varepsilon \rfloor) = B_{\lfloor n\varepsilon \rfloor}$ for all $k \geq \lfloor n\varepsilon \rfloor$ for large n. □

8 Finer Estimate on the Error and Proof of Lemma 15

In this section we will derive a finer estimate on the error $E_n(x, y)$ which was crucially used to prove Theorem 5. We look at each individual term of the series which appears in $E_n(x, y)$ and find a better bound than what we have before.

***Proof of Lemma* 15** Recall from (3.10)

$$E_n(x, y) = \lim_{t\to\infty} \mathbf{E}_x \left[\sum_{j=1}^{\eta_t} (-1)^{j-1} M_{j-1} \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1})\, \mathbb{1}_{[S_k = y]} \right].$$

Now conditioning on η_t we get

$$\begin{aligned} |E_n(x, y)| &= \left| \lim_{t\to\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{-t} \frac{t^\ell}{\ell!} \sum_{j=1}^{\ell} (-1)^{j-1} \mathbf{E}_x \left[M_{j-1} \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1})\, \mathbb{1}_{[S_k = y]} \right] \right| \\ &\leq \lim_{t\to\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{-t} \frac{t^\ell}{\ell!} \sum_{j=1}^{\ell} a_j, \end{aligned} \tag{8.1}$$

where

$$a_j := \mathbf{E}_x \left[M_{j-1} \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1})\, \mathbb{1}_{[S_k = y]} \right], \quad j \geq 1.$$

We now obtain bounds for a_j. First we bound each of the a_j's in terms of the distance of x and y from the boundary of V_n and then we bound the a_j's in terms of the distance $d(x, y)$.

Claim 19 For all $j \geq 1$ the following two estimates hold.

(a) **Bound in terms of distance from boundary.**

$$a_j \leq C_1(m) C_2(m) d(x, \partial_1 V_n) d(y, \partial_1 V_n) (m-1)^{-d(y, \partial_1 V_n)} \left(\frac{C_1(m)}{m} \right)^{j-1}. \tag{8.2}$$

(b) **Bound in terms of $d(x, y)$.**

$$a_j \leq (2j+1)^{4j+2} 3^{4j^2} \left(\frac{4(m-1)}{m-2} \right)^{2j+1} \frac{m-1}{m-2} (d(x, y))^{2j+1} (m-1)^{-d(x,y)}. \tag{8.3}$$

To obtain a finer estimate on $|E_n(x, y)|$ let us fix $J_0 \in [0, \infty)$ with the notion that when $J_0 = 0$, the sum $\sum_{j=1}^{J_0}$ is 0. Using the bounds (8.2) and (8.3) we have

$$\begin{aligned} \sum_{j=1}^{\ell} a_j &\leq \sum_{j=1}^{\infty} a_j = \sum_{j=1}^{J_0} a_j + \sum_{j=J_0+1}^{\infty} a_j \\ &\leq \sum_{j=1}^{J_0} (2j+1)^{4j+2} 3^{4j^2} \left(\frac{4(m-1)}{m-2} \right)^{2j+1} \frac{m-1}{m-2} (d(x, y))^{2j+1} (m-1)^{-d(x,y)} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=J_0+1}^{\infty} C_1(m)C_2(m)d(x,\partial_1 V_n)d(y,\partial_1 V_n)(m-1)^{-d(y,\partial_1 V_n)}\left(\frac{C_1(m)}{m}\right)^{j-1}\\
&\le J_0(2J_0+1)^{4J_0+2}3^{4J_0^2}\left(\frac{4(m-1)}{m-2}\right)^{2J_0+1}\frac{m-1}{m-2}(d(x,y))^{2J_0+1}(m-1)^{-d(x,y)}\\
&+C_1(m)C_2(m)\left(1-\frac{C_1(m)}{m}\right)^{-1} d(x,\partial_1 V_n)d(y,\partial_1 V_n)(m-1)^{-d(y,\partial_1 V_n)}\left(\frac{C_1(m)}{m}\right)^{J_0}.
\end{aligned}
$$

Here we have used the fact that $C_1(m)/m < 1$ for $m \ge 5$. Indeed, one can observe that the function $m \mapsto C_1(m)/m$ is a decreasing function and for $m = 4$ and 5 we compute that $C_1(4)/4 = 1.39$ and $C_1(5)/5 = 0.87$.

Now from (8.1) it follows that

$$
\begin{aligned}
|E_n(x,y)| \le\ & J_0(2J_0+1)^{4J_0+2}3^{4J_0^2}\left(\frac{4(m-1)}{m-2}\right)^{2J_0+1}\frac{m-1}{m-2}(d(x,y))^{2J_0+1}(m-1)^{-d(x,y)}\\
&+C_1(m)C_2(m)\left(1-\frac{C_1(m)}{m}\right)^{-1} d(x,\partial_1 V_n)d(y,\partial_1 V_n)(m-1)^{-d(y,\partial_1 V_n)}\left(\frac{C_1(m)}{m}\right)^{J_0}.
\end{aligned}
$$

From symmetry we conclude that (7.1) holds. We are now left to prove Claim 19.

***Proof of Claim* 19** First we prove part (a). The proof involves the successive use of the strong Markov property. We have

$$
\begin{aligned}
a_j &= \mathbf{E}_x\left[M_{j-1}\sum_{k=\tau_{j-1}}^{\tau_j-1}(k-\tau_{j-1})\,\mathbb{1}_{[S_k=y]}\right]\\
&= \mathbf{E}_x\left[\tau_0\mathbf{E}_{S_{\tau_0}}\left[\prod_{i=1}^{j-1}(\tau_i-\tau_{i-1}-1)\sum_{k=\tau_{j-1}}^{\tau_j-1}(k-\tau_{j-1})\,\mathbb{1}_{[S_k=y]}\right]\right]\\
&= \mathbf{E}_x\left[\tau_0\mathbf{E}_{S_{\tau_0}}\left[(\tau_1-1)\mathbf{E}_{S_{\tau_1}}\left[\prod_{i=1}^{j-2}(\tau_i-\tau_{i-1}-1)\sum_{k=\tau_{j-2}}^{\tau_{j-1}-1}(k-\tau_{j-2})\,\mathbb{1}_{[S_k=y]}\right]\right]\right].
\end{aligned}
$$

Iteratively using the strong Markov property we obtain

$$
a_j = \mathbf{E}_x\left[\underbrace{\tau_0\,\mathbf{E}_{S_{\tau_0}}\left[(\tau_1-1)\mathbf{E}_{S_{\tau_1}}\left[(\tau_1-1)\mathbf{E}_{S_{\tau_1}}\left[(\tau_1-1)\ldots\mathbf{E}_{S_{\tau_1}}\left[(\tau_1-1)\mathbf{E}_{S_{\tau_1}}\left[\sum_{k=0}^{\tau_1-1}k\,\mathbb{1}_{[S_k=y]}\right]\right]\right]\right]\right]}_{j-\text{ many expectations}}\right]. \tag{8.4}
$$

Note that for any $z \in \partial_1 V_n$

$$
\begin{aligned}
\mathbf{E}_z\,[\tau_1-1] &= \left[\mathbf{E}_z[\tau_1-1|S_1\in V_n]\mathbf{P}_z(S_1\in V_n)+\mathbf{E}_z[\tau_1-1|S_1\in V_n^c]\mathbf{P}_z(S_1\in V_n^c)\right]\\
&= \frac{1}{m}\mathbf{E}_z[\tau_1-1|S_1\in V_n] \overset{(6.1)}{\le} \frac{C_1(m)}{m}.
\end{aligned}
$$

This together with Lemma 14 and Lemma 13 gives the bound (8.2). □

Part (b)

We obtain a bound for a_j in terms of the distance between x and y. Let $p_k(z, w) = \mathbf{P}_z[S_k = w]$ be the k-step transition probability. We show it in two steps. First we show

$$a_j \leq \sum_{k=0}^{\infty} k^{2j+1} p_k(x, y) \tag{8.5}$$

and then we express $\sum_{k=0}^{\infty} k^{2j+1} p_k(x, y)$ in terms of the derivatives of $g(\mathbf{z}) = \Gamma(x, y|\mathbf{z}) = \sum_{k=0}^{\infty} \mathbf{P}_x [S_k = y] \mathbf{z}^k$. We explicitly compute these derivatives in Sect. 8.1.

We have from (8.4) that

$$\begin{aligned}
a_j &= \mathbf{E}_x \left[\underbrace{\tau_0\, \mathbf{E}_{S_{\tau_0}} \left[(\tau_1 - 1)\mathbf{E}_{S_{\tau_1}} \left[(\tau_1 - 1)\mathbf{E}_{S_{\tau_1}} \left[(\tau_1 - 1) \ldots \mathbf{E}_{S_{\tau_1}} \left[(\tau_1 - 1)\mathbf{E}_{S_{\tau_1}} \left[\sum_{k=0}^{\tau_1 - 1} k\, \mathbb{1}_{[S_k = y]} \right]\right]\right]\right]\right]}_{j-\text{ many expectations}} \right] \\
&= \sum_{z_1, \ldots, z_j \in \partial_1 V_n} \mathbf{E}_x \left[\tau_0\, \mathbb{1}_{[S_{\tau_0} = z_1]}\, \mathbf{E}_{z_1} \left[(\tau_1 - 1)\, \mathbb{1}_{[S_{\tau_1} = z_2]} \ldots \mathbf{E}_{z_{j-1}} \left[(\tau_1 - 1)\, \mathbb{1}_{[S_{\tau_1} = z_j]}\, \mathbf{E}_{z_j} \left[\sum_{k=0}^{\tau_1 - 1} k\, \mathbb{1}_{[S_k = y]} \right]\right]\right]\right] \\
&\leq \sum_{z_1, \ldots, z_j \in \partial_1 V_n} \sum_{\ell_0 = 0}^{\infty} \ell_0 p_{\ell_0}(z_j, y) \mathbf{E}_x \left[\tau_0\, \mathbb{1}_{[S_{\tau_0} = z_1]}\, \mathbf{E}_{z_1} \left[(\tau_1 - 1)\, \mathbb{1}_{[S_{\tau_1} = z_2]} \ldots \mathbf{E}_{z_{j-1}} \left[(\tau_1 - 1)\, \mathbb{1}_{[S_{\tau_1} = z_j]} \right]\right]\right] \\
&\leq \sum_{z_1, \ldots, z_j \in \partial_1 V_n} \sum_{\ell_0 = 0}^{\infty} \ell_0 p_{\ell_0}(z_j, y) \sum_{\ell_1 = 0}^{\infty} \ell_1 p_{\ell_1}(z_j, z_{j-1}) \ldots \sum_{\ell_j = 0}^{\infty} \ell_j p_{\ell_j}(x, z_1) \\
&\leq \sum_{\ell_0, \ell_1, \ldots \ell_j = 0}^{\infty} \ell_0 \ell_1 \ldots \ell_j p_{\ell_0 + \ell_1 + \ldots + \ell_j}(x, y) \\
&= \sum_{k=0}^{\infty} \sum_{\ell_1 = 0}^{k} \sum_{\ell_2 = 0}^{k - \ell_1} \cdots \sum_{\ell_j = 0}^{k - (\ell_1 + \ldots + \ell_{j-1})} (k - (\ell_1 + \ldots + \ell_j)) \ell_1 \ldots \ell_j p_k(x, y) \leq \sum_{k=0}^{\infty} k^{2j+1} p_k(x, y).
\end{aligned}$$

We now use the bound on the derivatives of g from (8.7) in (8.5) to obtain a bound for a_j in terms of $d(x, y)$. For that we first write k^{ℓ} in terms of $\prod_{i=0}^{i_0}(k - i)$, $i_0 = 0, 1, \ldots, \ell - 1$. We observe that

$$\begin{aligned}
k^2 &= k(k-1) + k, \\
k^3 &= k\,(k(k-1) + k) = \prod_{i=0}^{2}(k-i) + (2+1)\prod_{i=0}^{1}(k-i) + k, \\
k^4 &= k \left(\prod_{i=0}^{2}(k-i) + 3\prod_{i=0}^{1}(k-i) + k \right) \\
&= \prod_{i=0}^{3}(k-i) + (3+3)\prod_{i=0}^{2}(k-i) + (2 \times 3 + 1)\prod_{i=0}^{1}(k-i) + k.
\end{aligned}$$

In general we have that for any $k, \ell \geq 1$

$$k^{\ell} = \alpha_{\ell-1}^{(\ell)} \prod_{i=0}^{\ell-1}(k-i) + \alpha_{\ell-2}^{(\ell)} \prod_{i=0}^{\ell-2}(k-i) + \cdots + \alpha_1^{(\ell)} \prod_{i=0}^{1}(k-i) + \alpha_0^{(\ell)} k,$$

where the coefficients $\alpha_i^{(\ell)}$ for all $\ell \geq 1$ and $i = 0, 1, \ldots, \ell - 1$ are given recursively as follows

$$\alpha_0^{(\ell)} = \alpha_{\ell-1}^{(\ell)} = 1,$$
$$\alpha_i^{(\ell)} = (i+1)\alpha_i^{(\ell-1)} + \alpha_{i-1}^{(\ell-1)}, \quad 1 \leq i \leq \ell - 2.$$

It follows that for all $\ell \geq 1$ and $i = 0,\ 1,\ \ldots, \ell - 1$

$$\alpha_i^{(\ell)} \leq \ell! \leq \ell^\ell. \tag{8.6}$$

Now from (8.5) we have

$$\begin{aligned} a_j &\leq \sum_{k=0}^{\infty} k^{2j+1} p_k(x,y) \\ &= \sum_{k=0}^{\infty} p_k(x,y) \left[\alpha_{2j}^{(2j+1)} \prod_{i=0}^{2j}(k-i) + \alpha_{2j-1}^{(2j+1)} \prod_{i=0}^{2j-1}(k-i) + \cdots + \alpha_1^{(2j+1)} \prod_{i=0}^{1}(k-i) + \alpha_0^{(2j+1)} k \right] \\ &= \alpha_{2j}^{(2j+1)} g^{(2j+1)}(1) + \alpha_{2j-1}^{(2j+1)} g^{(2j)}(1) + \cdots + \alpha_1^{(2j+1)} g^{(2)}(1) + \alpha_0^{(2j+1)} g^{(1)}(1). \end{aligned}$$

Now using (8.6) and (8.7) we obtain

$$\begin{aligned} a_j &\leq (2j+1)(2j+1)^{2j+1} 3^{(2j)^2} (2j)^{2j} \left(\frac{4(m-1)}{m-2} \right)^{2j+1} \frac{m-1}{m-2} (d(x,y))^{2j+1} (m-1)^{-d(x,y)} \\ &\leq (2j+1)^{4j+2} 3^{4j^2} \left(\frac{4(m-1)}{m-2} \right)^{2j+1} \frac{m-1}{m-2} (d(x,y))^{2j+1} (m-1)^{-d(x,y)}. \end{aligned}$$

□

8.1 Bound on the Higher Derivatives of $\Gamma(x, y|\mathbf{z})$

In this section we obtain bound for the higher derivatives of the function $g(\mathbf{z}) = \Gamma(x, y|\mathbf{z})$ evaluated at the point $\mathbf{z} = 1$. Recall from (4.1) that for $x,\ y \in \mathbb{T}_m$

$$g(\mathbf{z}) = \Gamma(x, y|\mathbf{z}) = \frac{2(m-1)}{m-2+\sqrt{m^2-4(m-1)\mathbf{z}^2}} \left(\frac{m-\sqrt{m^2-4(m-1)\mathbf{z}^2}}{2(m-1)\mathbf{z}} \right)^{d(x,y)}, \quad \mathbf{z} \in \mathbb{C}.$$

We prove the following bound.

Lemma 20 *Let $x, y \in \mathbb{T}_m$ and $d = d(x, y)$. Then for $k \geq 1$*

$$g^{(k)}(1) \leq 3^{(k-1)^2} (k-1)^{k-1} \left(\frac{4(m-1)}{m-2} \right)^k \frac{m-1}{m-2} d^k (m-1)^{-d}, \tag{8.7}$$

and

$$g(1) = \frac{m-1}{m-2}(m-1)^{-d}.$$

Proof We write $\rho(\mathbf{z}) := \sqrt{m^2 - 4(m-1)\mathbf{z}^2}$. Then

$$g(\mathbf{z}) = \frac{2(m-1)}{m-2+\rho(\mathbf{z})} \left(\frac{m-\rho(\mathbf{z})}{2(m-1)\mathbf{z}} \right)^d. \tag{8.8}$$

We have

$$g(1)=\frac{m-1}{m-2}(m-1)^{-d}.$$

Taking logarithms on both sides of (8.8) and then differentiating we get

$$\begin{aligned}\frac{g'(\mathbf{z})}{g(\mathbf{z})}&=\frac{4(m-1)\mathbf{z}}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{d4(m-1)\mathbf{z}}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})}-\frac{d}{\mathbf{z}}\\&=:h(\mathbf{z}).\end{aligned}$$

Note that here we have used $\rho'(\mathbf{z})=-(4(m-1)\mathbf{z})/\rho(\mathbf{z})$. So we have

$$g'(\mathbf{z})=g(\mathbf{z})h(\mathbf{z}).$$

To obtain bounds for the derivatives of g we first bound h and its derivatives evaluated at $\mathbf{z}=1$. We have

$$h(1)=\frac{4(m-1)}{2(m-2)^2}+\frac{d4(m-1)}{2(m-2)}-d\leq\frac{4(m-1)}{m-2}d.$$

Differentiating h we get

$$\begin{aligned}h'(\mathbf{z})=&\frac{4(m-1)}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})}\left[1-\frac{\mathbf{z}\rho'(\mathbf{z})}{m-2+\rho(\mathbf{z})}-\frac{\mathbf{z}\rho'(\mathbf{z})}{\rho(\mathbf{z})}\right]\\&+\frac{d4(m-1)}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})}\left[1+\frac{\mathbf{z}\rho'(\mathbf{z})}{m-\rho(\mathbf{z})}-\frac{\mathbf{z}\rho'(\mathbf{z})}{\rho(\mathbf{z})}\right]-\frac{d}{\mathbf{z}^2}.\end{aligned}$$

Note that $\rho(1)=m-2$ and $\rho'(1)=-(4(m-1))/(m-2)$. Using these values we have

$$\begin{aligned}h'(1)=&\frac{4(m-1)}{2(m-2)^2}\left[1+\frac{4(m-1)}{2(m-2)^2}+\frac{4(m-1)}{(m-2)^2}\right]\\&+\frac{d4(m-1)}{2(m-2)}\left[1-\frac{4(m-1)}{2(m-2)}+\frac{4(m-1)}{(m-2)^2}\right]-d\\\leq&\,3\frac{4(m-1)}{m-2}\left[\frac{4(m-1)}{2(m-2)^2}+\frac{d4(m-1)}{2(m-2)}\right]\\\leq&\,3\left(\frac{4(m-1)}{m-2}\right)^2d.\end{aligned}\tag{8.9}$$

To obtain a bound on $h''(1)$ we write $h'(\mathbf{z})$ as

$$\begin{aligned}h'(\mathbf{z})=&\frac{4(m-1)}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{(4(m-1))^2\mathbf{z}^2}{(m-2+\rho(\mathbf{z}))^2\rho(\mathbf{z})^2}+\frac{(4(m-1))^2\mathbf{z}^2}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})^3}\\&+\frac{d4(m-1)}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})}-\frac{d(4(m-1))^2\mathbf{z}^2}{(m-\rho(\mathbf{z}))^2\rho(\mathbf{z})^2}+\frac{d(4(m-1))^2\mathbf{z}^2}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})^3}-\frac{d}{\mathbf{z}^2}.\end{aligned}$$

Now differentiating with respect to $\mathbf{z}$ we obtain

$$\begin{aligned}h''(\mathbf{z})=&\frac{4(m-1)}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})}\left[\frac{4(m-1)\mathbf{z}}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{4(m-1)\mathbf{z}}{\rho(\mathbf{z})^2}\right]\\&+\frac{(4(m-1))^2}{(m-2+\rho(\mathbf{z}))^2\rho(\mathbf{z})^2}\left[2\mathbf{z}+\frac{2(4(m-1))\mathbf{z}^3}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{2(4(m-1))\mathbf{z}^3}{\rho(\mathbf{z})^2}\right]\\&+\frac{(4(m-1))^2}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})^3}\left[2\mathbf{z}+\frac{(4(m-1))\mathbf{z}^3}{(m-2+\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{3(4(m-1))\mathbf{z}^3}{\rho(\mathbf{z})^2}\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{d4(m-1)}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})}\left[-\frac{4(m-1)\mathbf{z}}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{4(m-1)\mathbf{z}}{\rho(\mathbf{z})^2}\right]\\
&-\frac{d(4(m-1))^2}{(m-\rho(\mathbf{z}))^2\rho(\mathbf{z})^2}\left[2\mathbf{z}-\frac{2(4(m-1))\mathbf{z}^3}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{2(4(m-1))\mathbf{z}^3}{\rho(\mathbf{z})^2}\right]\\
&+\frac{d(4(m-1))^2}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})^3}\left[2\mathbf{z}-\frac{4(m-1)\mathbf{z}^3}{(m-\rho(\mathbf{z}))\rho(\mathbf{z})}+\frac{3(4(m-1))\mathbf{z}^3}{\rho(\mathbf{z})^2}\right]+\frac{2d}{\mathbf{z}^3}.
\end{aligned}
$$

Putting $\mathbf{z}=1$ we have

$$
\begin{aligned}
h''(1)=&\frac{4(m-1)}{2(m-2)^2}\left[\frac{4(m-1)}{2(m-2)^2}+\frac{4(m-1)}{(m-2)^2}\right]\\
&+\frac{(4(m-1))^2}{4(m-2)^4}\left[2+\frac{2(4(m-1))}{2(m-2)^2}+\frac{2(4(m-1))}{(m-2)^2}\right]\\
&+\frac{(4(m-1))^2}{2(m-2)^4}\left[2+\frac{(4(m-1))}{2(m-2)^2}+\frac{3(4(m-1))}{(m-2)^2}\right]\\
&+\frac{d4(m-1)}{2(m-2)}\left[-\frac{4(m-1)}{2(m-2)}+\frac{4(m-1)}{(m-2)^2}\right]\\
&-\frac{d(4(m-1))^2}{4(m-2)^2}\left[2-\frac{2(4(m-1))}{2(m-2)}+\frac{2(4(m-1))}{(m-2)^2}\right]\\
&+\frac{d(4(m-1))^2}{2(m-2)^3}\left[2-\frac{4(m-1)}{2(m-2)}+\frac{3(4(m-1))}{(m-2)^2}\right]+2d.
\end{aligned}
$$

We observe that the term inside the square bracket in each summand is bounded by $(9(4(m-1)))/(m-2)$ and the other terms are the same as the summands in $h'(1)$ except for the last term. So we conclude using (8.9) that

$$
h''(1)\le 9\cdot 3\left(\frac{4(m-1)}{m-2}\right)^3 d\le 9\cdot 4\left(\frac{4(m-1)}{m-2}\right)^3 d.
$$

In a similar way we can write $h^{(k)}(1)$ so that the term inside the square bracket in each summand is bounded by $(3(2k-1)(4(m-1)))/(m-2)$ and the other terms are the same as the summands in $h^{(k-1)}(1)$ except the last term. Hence we conclude that

$$
\begin{aligned}
h^{(k)}(1)&\le 3^k(1\cdot 3\cdot 5\cdots(2k-1))\left(\frac{4(m-1)}{m-2}\right)^{(k+1)} d\\
&\le 3^k k^k\left(\frac{4(m-1)}{m-2}\right)^{(k+1)} d,
\end{aligned}
$$

where we obtain the second inequality by using the relation between the arithmetic and the geometric mean. We now prove (8.7) by the method of induction. We have

$$
g^{(1)}(1)=g(1)h(1)\le\frac{m-1}{m-2}(m-1)^{-d}\frac{4(m-1)}{m-2}d.
$$

Assume that (8.7) holds true for $k=1,\ldots,\ell-1$. Now we have

$$
\begin{aligned}
g^{(\ell)}(1)&=(gh)^{(\ell-1)}(1)=\sum_{k=0}^{\ell-1}\binom{\ell-1}{k}g^{(\ell-1-k)}(1)h^{(k)}(1)\\
&\le\sum_{k=0}^{\ell-2}\binom{\ell-1}{k}\left[3^{(\ell-1-k-1)^2}(\ell-1-k-1)^{\ell-1-k-1}\left(\frac{4(m-1)}{m-2}\right)^{\ell-1-k}\frac{m-1}{m-2}d^{\ell-1-k}\right.
\end{aligned}
$$

$$
\begin{aligned}
&(m-1)^{-d}\Big] 3^k k^k \left(\frac{4(m-1)}{m-2}\right)^{(k+1)} d + \frac{m-1}{m-2} d(m-1)^{-d} 3^{\ell-1}(\ell-1)^{\ell-1}\left(\frac{4(m-1)}{m-2}\right)^{\ell} d \\
&\le \left[3^{(\ell-2)^2}(\ell-1)^{\ell-1}\left(\frac{4(m-1)}{m-2}\right)^{\ell}\frac{m-1}{m-2}d^{\ell}(m-1)^{-d}\right]\sum_{k=0}^{\ell-1}\binom{\ell-1}{k} \\
&= \left[3^{(\ell-2)^2}(\ell-1)^{\ell-1}\left(\frac{4(m-1)}{m-2}\right)^{\ell}\frac{m-1}{m-2}d^{\ell}(m-1)^{-d}\right]2^{\ell-1} \\
&\le 3^{(\ell-1)^2}(\ell-1)^{\ell-1}\left(\frac{4(m-1)}{m-2}\right)^{\ell}\frac{m-1}{m-2}d^{\ell}(m-1)^{-d}.
\end{aligned}
$$

Therefore by induction (8.7) holds for all $k \geq 1$. □

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Appendix A: An Alternative Argument for (4.6)

After the revision of the paper it was pointed out to us by an anonymous referee that an alternative proof can be carried out to quantitatively estimate the error one commits by replacing G_n by G. This proof gives a bound comparable to (4.6) for points that are far away from the boundary. For completeness we would like to outline this proof here.

Proof The proof is based on a double application of the maximum principle for harmonic functions [3, Theorem 1.37]. Fix $y \in V_n$. We define the function $H_y(\cdot)$ as

$$
\begin{aligned}
V_n &\to \mathbb{R} \\
x &\mapsto G_n(x, y) - G(x, y).
\end{aligned}
$$

We then set

$$
a := \sup_{x \in V_{n-1}^{c}} \left| H_y(x) \right| .
$$

We have that

$$\begin{cases} \left|\Delta H_y(x)\right| \leq 2a & x \in V_n^{\mathrm{c}} \\ \Delta(\Delta H_y)(x) = 0 & x \in V_n \end{cases}$$

so that $\Delta H_y(\cdot)$ is harmonic in V_n. We can invoke the the maximum principle to say that $\max_{x\in V_n} \left|\Delta H_y(x)\right| \leq 2a$. Now consider the function

$$f(x) := a + \frac{2am}{m-2} d(x,\, \partial_1 V_n).$$

It is clear that $f(x) = a$ on V_n^{c}, and moreover that for $x \neq o$

$$\begin{aligned} \Delta f(x) &= \frac{2a}{m-2}\left[(m-1)\left(d(x,\,\partial_1 V_n) - 1\right) + \left(d(x,\,\partial_1 V_n) + 1\right) - md(x,\,\partial_1 V_n)\right] \\ &= \frac{2a}{m-2}(2-m) = -2a \end{aligned}$$

while for $x = o$ we have $\Delta f(x) = -2am/(m-2) \leq -2a$. So the function

$$H_y(\cdot) - f(\cdot)$$

is subharmonic in V_n and again by the maximum principle

$$\max_{x\in V_n} |H(\cdot) - f(\cdot)| = \max_{x\in V_{n+1}\setminus V_n} H_y(\cdot) - f(\cdot) \leq 0$$

since $\left|H_y(x)\right| \leq a = f(x)$ by the definition of a for $x \in V_{n+1} \setminus V_n$. Running the same argument for $-f$ rather than f we finally obtain that $|H_n(x)| \leq f(x)$ in V_n.

This implies that for $x \in V_n$

$$\begin{aligned} |H_n(x)| = |G_n(x, y) - G(x, y)| &\leq a\left(1 + C(m)d(x,\,\partial_1 V_n)\right) \\ &\leq C'(m)d(x,\,\partial_1 V_n) \sup_{x'\in V_{n-1}^{\mathrm{c}}} \left|G(x', y)\right| \\ &\leq \frac{C'(m)d(x,\,\partial_1 V_n)d(y,\,\partial_1 V_n)}{(m-1)^{d(y,\,\partial_1 V_n)}}. \end{aligned}$$

Being our argument symmetric in x and y, we can conclude our result. □

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